0. Vectors, matrices, and differential equations

Bring solutions along to the class at 10am on 5/10/20.

Most of these questions are intended as a review of first-year topics, but some new content will be introduced, indicated by *.

0.1 * Linear vector spaces

You will have seen various methods of writing vectors, such as $\boldsymbol{v}$, $\underline{v}$, $\overrightarrow{v}$. A particularly convenient notation for complex vectors, which we will use in this course, is $|v\rangle$. Define the N-dimensional vector $|v\rangle$ by

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \end{pmatrix}. \tag{1}$$

Note that an N-dimensional vector is an $N \times 1$ matrix.

0.1.1 *

Look up the axioms defining a linear vector space. State them here, using this new notation. Use this new notation in the following questions.

0.2 Matrix products

0.2.1

For the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

show that the eigenvalues are φ and φ^{-1} where $\varphi = \left(1+\sqrt{5}\right)/2$ is the golden ratio. Find the normalized eigenvectors.

0.2.2

Show that

$$M^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3}$$

0.2.3

Prove by induction that

$$M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{4}$$

for $n > 1$, where

$$\begin{aligned} F_n &= 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots \\ n &= 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots \end{aligned} \tag{5}$$

are the Fibonacci numbers defined by $F_{n+1} = F_n + F_{n-1}$ for $n > 1$.

0.3 Matrix transpose

For two square matrices A and B prove that

$$(AB)^T = B^T A^T. \tag{6}$$

0.4 Eigenvalues and eigenvectors

The Pauli matrices are defined to be

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

0.4.1

Find the eigenvalues and eigenvectors of each.

0.4.2

Show that each matrix squares to the identity.

0.4.3

Show that each matrix is its own inverse.

0.4.4 *

The ‘commutator’ of two matrices A and B is defined as

$$[A, B] \triangleq AB - BA \tag{8}$$

where $\triangleq$ indicates that this is a definition. Show that

$$\left[\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y\right] = i\frac{1}{2}\sigma_z. \tag{9}$$

0.5 Determinants

0.5.1

List the rules for simplifying determinants.

0.5.2

Show that the determinant of the matrix

$$M = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & \theta & -\theta & 0 \\ 0 & e^{ix} & e^{-ix} & -1 \\ 0 & \theta e^{ix} & -\theta e^{-ix} & -1 \end{pmatrix} \tag{10}$$

is

$$\det(M) = 4\theta\cos(x) - 2i\left(1+\theta^2\right)\sin(x). \tag{11}$$

This is used when proving a result you will see when studying quantum tunnelling.

0.5.3

Find element $(1,4)$ of the inverse matrix M^{-1}, *i.e.* M_{14}^{-1}. Note that you do not need to compute the entire inverse matrix to do this.

0.6 Separable PDEs

The one-dimensional wave equation is a partial differential equation (PDE):

$$\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)_x = c^2\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)_t \tag{12}$$

where the subscripts indicate what is held constant in each case.

0.6.1

Substituting the ansatz

$$u(x,t) = \phi(x)\,T(t) \tag{13}$$

show that the equation can be separated into two ordinary differential equations (ODEs):

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} = \frac{1}{\phi(x)}\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2}. \tag{14}$$

0.6.2

Since both sides are equal for all t and x they must be equal to the same constant. Calling this constant $-\lambda^2$ show that general solutions take the form

$$T(t) = A\sin(c\lambda t) + B\cos(c\lambda t) \tag{15}$$

$$\phi(x) = C\sin(\lambda x) + D\cos(\lambda x). \tag{16}$$

0.6.3

Say the equation is describing the motion of a string fixed at 0 and L. This implies the boundary conditions

$$\phi(0) = \phi(L) = 0. \tag{17}$$

Explain why this implies

(a) $D = 0$

(b) $\lambda = n\pi/L$.

0.6.4

Say the string is initially at rest, *i.e.*

$$\left(\frac{\partial u(x,t)}{\partial t}\right)_x\bigg|_{t=0} = \dot{u}(x,0) = 0. \tag{18}$$

Show that the solutions take the form

$$u_n(x,t) = a_n\cos\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \tag{19}$$

where a_n is a normalization constant. Sketch the two lowest-energy solutions at a few key times.

0.7 * Differential operators

Consider the partial derivative

$$\frac{\partial u(x,t)}{\partial t} \tag{20}$$

where the explicit statement as to which variables are held constant (*i.e.* x) has been omitted. A convenient trick is to consider the partial derivative to be an 'operator' $\partial/\partial t$ acting on the function $u(x,t)$:

$$\left(\frac{\partial}{\partial t}\right)u(x,t) \triangleq \frac{\partial u(x,t)}{\partial t}. \tag{21}$$

This can be abbreviated even further to

$$\partial_t u(x,t). \tag{22}$$

The utility of this notation derives from being able to write expressions such as

$$\left(\frac{\partial}{\partial t}\right)^2 = (\partial_t)^2 = \partial_t \partial_t = \partial_t^2. \tag{23}$$

You can always imagine the operator to be acting on some arbitrary function, as in Eq. 21, but it is often easier to work with the operators themselves. You have probably seen some differential operators before. For example, you may have seen the Laplace operator (Laplacian)

$$\triangle \triangleq \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{24}$$

or the d'Alembert operator (d'Alembertian):

$$\square \triangleq c^{-2}\partial_t^2 - \triangle = c^{-2}\partial_t^2 - \nabla^2. \tag{25}$$

0.7.1 *

Show that

$$\square u = 0 \tag{26}$$

is a succinct way of writing the 3D wave equation.

0.7.2 *

By expanding the product show that the 1D wave equation can be written as

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)\, u(x,t) = 0. \tag{27}$$

0.7.3 *

Show that the 1D wave equation can also be written as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)\, u(x,t) = 0. \tag{28}$$

0.7.4 *

Operators, like matrices, may also fail to commute. Using the definition of the commutator in Eq. 8, along with Eqs. 27 and 28, deduce that the operators $\partial_t + c\partial_x$ and $\partial_t - c\partial_x$ commute:

$$[(\partial_t + c\partial_x), (\partial_t - c\partial_x)]\, u(x,t) = 0 \quad \forall \quad u. \tag{29}$$

0.7.5 *

Define the variables $\eta = x - ct$ and $\nu = x + ct$.

Recall the chain rule:

$$\left(\frac{\partial f}{\partial \nu}\right)_\eta = \left(\frac{\partial x}{\partial \nu}\right)_\eta \left(\frac{\partial f}{\partial x}\right)_t + \left(\frac{\partial t}{\partial \nu}\right)_\eta \left(\frac{\partial f}{\partial t}\right)_x \tag{30}$$

which can be written as the following operators acting on f:

$$\partial_\nu = (\partial_\nu x)\,\partial_x + (\partial_\nu t)\,\partial_t. \tag{31}$$

(a) Show that

$$2c\partial_\eta = c\partial_x - \partial_t \tag{32}$$

$$2c\partial_\nu = c\partial_x + \partial_t. \tag{33}$$

(b) Therefore show that the wave equation in these co-ordinates reads

$$\partial_\eta \partial_\nu u\,(\eta, \nu) = 0. \tag{34}$$

(c) Deduce that a general solution to the wave equation can be written as

$$u\,(\eta, \nu) = f\,(\eta) + g\,(\nu) \tag{35}$$

or

$$u\,(x, t) = f\,(x - ct) + g\,(x + ct) \tag{36}$$

for arbitrary functions f and g.

(d) Interpret the result in terms of travelling waves.