2 Scattering and tunnelling

Bring solutions along to the class at 10am on 12/10/20.

Questions marked # are beyond the syllabus. We may discuss them in the class if there is time. Attempt all questions. Feel free to discuss with friends. Don't worry if you cannot answer a question fully as you will discuss the solutions in the class.

2.0 Videos

Please watch this week's videos: V2.1a, V2.1b, V2.1c, V2.1d, V2.2, V2.3
All videos are available on the youtube channel Introductory Quantum Mechanics (link available at felixflicker.com/teaching, or on Learning Central).

2.1 Scattering from a potential step

A particle is incident from the left ($x = -\infty$) on a potential step defined by

$$V(x) = \begin{cases} V_0,\, x \geq 0\,(\text{region } I) \\ 0,\, x < 0\,(\text{region } II). \end{cases} \tag{44}$$

2.1.1

Assuming $E < V_0$ explain why the solutions to the TISE take the forms

$$\phi_I(x) = \exp(ikx) + r\exp(-ikx) \tag{45}$$

$$\phi_{II}(x) = t\exp(-\kappa x). \tag{46}$$

What are the corresponding forms for $E > V_0$?

2.1.2

Find expressions for k and κ in terms of E and V_0, and the equivalent for $E < V_0$.

2.1.3

State the two boundary conditions obeyed at the step.

2.1.4

Find the reflection and transmission amplitudes for the cases
(a) $E < V_0$
(b) $E > V_0$.

2.1.5

Using the probability current densities find the reflection (R) and transmission (T) probabilities for both cases.

2.1.6

Show that $R + T = 1$ in both cases.

2.1.7

Show that if the step occurs at $x = a$ instead of $x = 0$ the reflection and transmission amplitudes change, but the reflection and transmission probabilities remain the same.

2.1.8

Sketch the waves in each region, paying attention to the amplitude, phase, and wavelength of the waves, for the following cases:
(a) $E \gg V_0$
(b) $E \gtrsim V_0$
(c) $E \lesssim V_0$
(d) $E \approx 0$.

2.1.9

Derive the transmission and reflection amplitudes and probabilities assuming instead the potential

$$V(x) = \begin{cases} V_0,\ x < 0\,(\text{region } 1) \\ 0,\ x \geq 0\,(\text{region } 2) \end{cases} \tag{47}$$

but with the wave still incident from the left ($x = -\infty$). Explain why we require $E > V_0$ in this case.

2.2 Scattering over a barrier

Consider the potential

$$V(x) = \begin{cases} 0,\ x < -L\,(\text{region } 1) \\ V_0,\ -L \leq x \leq L\,(\text{region } 2) \\ 0,\ x \geq L\,(\text{region } 3). \end{cases} \tag{48}$$

Assume $E > V_0$, and that a wave is incident from the left ($x = -\infty$). Define the solutions in each region to be

$$\phi_1 = \exp(ikx) + r\exp(-ikx) \tag{49}$$

$$\phi_2 = a\exp(ik'x) + b\exp(-ik'x) \tag{50}$$

$$\phi_3 = t\exp(-ikx). \tag{51}$$

2.2.1

Sketch the potential.

2.2.2

Identify the conditions on k and k' in terms of E and V_0.

2.2.3

State the two boundary conditions at each end of the barrier.

2.2.4

(a) Show that the four boundary conditions lead to the matrix equation

$$\begin{pmatrix} -\exp(ikL) & \exp(-ik'L) & \exp(ik'L) & 0 \\ k\exp(ikL) & k'\exp(-ik'L) & -k'\exp(ik'L) & 0 \\ 0 & \exp(ik'L) & \exp(-ik'L) & -\exp(ikL) \\ 0 & k'\exp(ik'L) & -k'\exp(-ik'L) & -k\exp(ikL) \end{pmatrix} \begin{pmatrix} r \\ a \\ b \\ t \end{pmatrix} = \begin{pmatrix} \exp(-ikL) \\ k\exp(-ikL) \\ 0 \\ 0 \end{pmatrix}. \tag{52}$$

(b) Calling the matrix M, show that

$$t = \exp(-ikL)\left(M^{-1}_{41} + kM^{-1}_{42}\right) = \frac{\exp(-ikL)}{\det(M)}\left(C^{-1}_{14} + kC^{-1}_{24}\right) \tag{53}$$

where C is the matrix of signed cofactors.

2.2.5

The resonant transmission condition is that $k'L = n\pi$ for integer n.

(a) For which energies is the barrier at resonance?

(b) Considering the case of n even, show that Eqn. 52 can be written

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ k & k' & -k' & 0 \\ 0 & 1 & 1 & -1 \\ 0 & k' & -k' & -k \end{pmatrix} \begin{pmatrix} r\exp(ikL) \\ a \\ b \\ t\exp(ikL) \end{pmatrix} = \begin{pmatrix} \exp(-ikL) \\ k\exp(-ikL) \\ 0 \\ 0 \end{pmatrix}. \tag{54}$$

(c) Therefore, using the rules for simplifying determinants, show that the probability of transmission is indeed one, and that of reflection is zero.

2.2.6 Scattering over a finite potential well

Switching the sign of V_0 in Eq. 48 gives the potential of a finite well. Later in the course we will see that this is able to trap bound states for $E < 0$. However, if $E > 0$ the solutions are still plane waves. For what values of the energy will a plane wave pass unhindered through the well?

2.3 Quantum tunnelling

Return to the potential of Eq. 48 in 2.4. Now consider the case $E < V_0$.

2.3.1

What are the classical probabilities for transmission and reflection?

2.3.2

Explain why the only change to the form of the wavefunctions is to change $k' \to -i\kappa$ in Eq. 50.

2.3.3

Identify the condition on κ in terms of E and V_0.

2.3.4

Deduce that the matrix equation, Eq. 52, changes to

$$\begin{pmatrix} -\exp(ikL) & \exp(-\kappa L) & \exp(\kappa L) & 0 \\ k\exp(ikL) & -i\kappa\exp(-\kappa L) & i\kappa\exp(\kappa L) & 0 \\ 0 & \exp(\kappa L) & \exp(-\kappa L) & -\exp(ikL) \\ 0 & -i\kappa\exp(\kappa L) & i\kappa\exp(-\kappa L) & -k\exp(ikL) \end{pmatrix} \begin{pmatrix} r \\ a \\ b \\ t \end{pmatrix} = \begin{pmatrix} \exp(-ikL) \\ k\exp(-ikL) \\ 0 \\ 0 \end{pmatrix}. \tag{55}$$

2.3.5

Is resonant transmission possible?

2.3.6

Show that the probability of transmission through the barrier is

$$T = \frac{4k^2\kappa^2}{4k^2\kappa^2 + (\kappa^2 + k^2)^2 \sinh^2(2\kappa L)}. \tag{56}$$

HINT: use Eq. 53 and the rules for simplifying determinants.

2.3.7

Hence show that

$$T = \frac{4E\left(V_0 - E\right)}{4E\left(V_0 - E\right) + V_0^2 \sinh^2\left(2\sqrt{2m\left(V_0 - E\right)}L/\hbar\right)}.$$

Does this make sense with regard to the barrier penetration problem shown in the notes and lectures?