3 Bound states (I)

Bring solutions along to the class at 10am on 19/10/20.

Questions marked # are beyond the syllabus. We may discuss them in the class if there is time. Attempt all questions. Feel free to discuss with friends. Don't worry if you cannot answer a question fully as you will discuss the solutions in the class.

3.0 Videos

Please watch this week's videos: V3.1, V3.2, V3.3, V3.4, V3.5
All videos are available on the youtube channel Introductory Quantum Mechanics (link available at felixflicker.com/teaching, or on Learning Central).

3.1 The infinite potential well

Consider the TISE

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right)\phi_n(x) = E_n\phi_n(x) \tag{55}$$

with the potential

$$V(x) = \begin{cases} 0, & -L/2 \le x \le L/2 \\ \infty, & \text{otherwise.} \end{cases} \tag{56}$$

3.1.1

Sketch the potential and the first five energy eigenfunctions.

3.1.2

Which eigenfunctions $\phi_n(x)$ will be odd, and which even? Bearing this in mind will help simplify many of the remaining questions!

3.1.3

Find the eigenvalues E_n and normalized eigenfunctions $\phi_n(x)$. Note that you will get different forms for the odd and even functions. Why do you not need to worry about the complex phase?

3.1.4

Show that all eigenfunctions are orthonormal.

3.1.5

Write down the solutions to the TDSE, $\psi_n(x,t)$.

3.1.6

Show that the probability density

$$\rho(x) = |\psi_n(x,t)|^2 \tag{57}$$

is time independent for all energy eigenfunctions.

3.1.7

Say the particle in the well is known to have energy E_3. What is the probability to find the particle in the leftmost quarter of the well?

3.1.8

Explain, without detailed calculation, why the expectation values of position, defined as:

$$\langle \hat{x} \rangle \triangleq \int_{-\infty}^{\infty} \psi_n^*(x,t)\, x \psi_n(x,t)\, \mathrm{d}x \tag{58}$$

and momentum, defined as:

$$\langle \hat{p} \rangle \triangleq \int_{-\infty}^{\infty} \psi_n^*(x,t)\, (-i\hbar\partial_x)\, \psi_n(x,t)\, \mathrm{d}x \tag{59}$$

must be zero for any energy eigenfunction in the well.

3.1.9

At which positions are you most likely to find a particle described by eigenstate ψ_n? Explain why there is no contradition with the previous result.

3.1.10

(a) The expected value of x^2 for eigenstate ψ_n is

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} \psi_n^*(x,t)\, x^2 \psi_n(x,t)\, \mathrm{d}x. \tag{60}$$

Show that

$$\langle \hat{x}^2 \rangle = \frac{L^2}{12} + \frac{L^2}{2n^2\pi^2}. \tag{61}$$

(b) The expected value of p^2 for eigenstate ψ_n is

$$\langle \hat{p}^2 \rangle = \int_{-\infty}^{\infty} \psi_n^*(x,t)\, (-i\hbar\partial_x)^2\, \psi_n(x,t)\, \mathrm{d}x. \tag{62}$$

Show that

$$\langle \hat{p}^2 \rangle = \left(\frac{\hbar n \pi}{L} \right)^2 . \tag{63}$$

HINT: one method is to use the TISE.

(c) The Heisenberg uncertainty principle states that

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{64}$$

where

$$\sigma_{\hat{A}} \triangleq \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} \tag{65}$$

is the uncertainty (standard deviation). Show that the uncertainty principle is obeyed by the eigenstates of the infinite well.

3.2 Infinite sets of bound states form an orthonormal basis

Whenever all the solutions of the TISE are bound states $\phi_n(x)$, these states must form an orthonormal basis:

$$\int_{-\infty}^{\infty} \phi_n^*(x)\, \phi_m(x)\, \mathrm{d}x = \delta_{nm} \tag{66}$$

where the Kronecker delta is defined as

$$\delta_{nm} = \begin{cases} 1, \; n = m \\ 0, \text{ otherwise.} \end{cases}$$

Show that this implies that any function can be written as a sum of these energy eigenstates

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \tag{67}$$

and identify the (possibly complex) coefficients f_n.