4 Bound states (II)

Bring solutions along to the class at 10am on 26/11/20.

Questions marked # are beyond the syllabus. We may discuss them in the class if there is time. Attempt all questions. Feel free to discuss with friends. Don't worry if you cannot answer a question fully as you will discuss the solutions in the class.

4.0 Videos

Please watch this week's videos: V4.1, V4.2

All videos are available on the youtube channel Introductory Quantum Mechanics (link available at felixflicker.com/teaching, or on Learning Central).

4.1 Linearity of quantum mechanics

Show that if

$$\psi_n(x,t) = \phi_n(x)\exp(-iE_n t/\hbar) \tag{70}$$

is a solution to the TDSE, then any superposition

$$f(x,t) = \sum_{n=1}^{\infty} f_n \psi_n(x,t) \tag{71}$$

is also a solution.

4.2 Superposition of two energy eigenstates

Consider again the infinite well potential of Eq. 58, and the energy eigenvalues and normalized eigenvectors you found in question **3.1.3**.

4.2.1

Consider the state

$$\chi(x,t) = \mathcal{N}\left(3\psi_1(x,t) + 4\psi_2(x,t)\right). \tag{72}$$

which is a quantum superposition of two energy eigenstates. What is the value of the normalization constant $\mathcal{N}$?

4.2.2

(a) What are the possible outcomes of a measurement of the energy of state χ?

(b) What are the probabilities of each outcome being found?

4.2.3

Find the time-dependent probability density of $\chi(x,t)$. After how long does the state return to its original form?

4.2.4

Show that the expected value of position for the state $\chi(x,t)$ is

$$\langle x \rangle_\chi = \left(\frac{L}{2}\right) \cdot \frac{2^8}{75\pi^2} \cos\left(\frac{3\hbar\pi^2 t}{2mL^2}\right).$$

HINT:

$$\cos(Ax)\sin(2Ax) = 2\cos^2(Ax)\sin(Ax) \tag{73}$$

$$= -\frac{2}{3A}\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^3(Ax)\right) \tag{74}$$

$$= -\frac{1}{6A}\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(3Ax) + 3\cos(Ax)\right) \tag{75}$$

(check you can derive each step!).

4.3 Deriving the time evolution of a given state

Consider again the infinite well potential of Eq. 58, and the energy eigenvalues and normalized eigenvectors you found in question **3.1.3**. A state is prepared in the well which has a wavefunction given by a top hat function:

$$\varpi(x) = \begin{cases} (\alpha L)^{-1/2}, & -\alpha\frac{L}{2} \le x \le \alpha\frac{L}{2} \\ 0, & \text{otherwise} \end{cases} \tag{76}$$

where $0 < \alpha \le 1$.

4.3.1

Check this state is normalised.

4.3.2

Show that this state can be written in terms of energy eigenstates as:

$$\varpi(x) = \sqrt{2\alpha} \sum_{n \text{ odd}} \operatorname{sinc}\left(\frac{n\pi\alpha}{2}\right) \phi_n(x). \tag{77}$$

4.3.3

State the subsequent time evolution.

4.3.4

Using the known result

$$\sum_{n \text{ odd}}^{\infty} \text{sinc}^2(n) = \frac{\pi}{4} \tag{78}$$

show explicitly that $\varpi(x)$ in Eq. 77 is normalised for the case $\alpha = 2/\pi$.

4.3.5

Given that we already know the state must be normalised, derive an expression for

$$\sum_{n \text{ odd}}^{\infty} \text{sinc}^2(\beta n) \tag{79}$$

for arbitrary real β.

4.3.6

By considering the limit $\alpha \to 0$, explain why an observation of the particle at the centre of the well implies the particle is equally likely to be observed in the left half of the well as the right for all subseqent times.

4.4 The finite potential well

Consider the potential

$$V(x) = \begin{cases} 0, \ x \leq -L & \text{(region 1)} \\ -V_0, \ -L < x < L & \text{(region 2)} \\ 0, \ L \leq x & \text{(region 3)} \end{cases} \tag{80}$$

and denote the solutions to the TISE in each region $\phi_i(x)$ with $i \in [1, 3]$.

4.4.1

Sketch the potential and three bound states, assuming three exist.

4.4.2

State the two boundary conditions at each end of the well.

4.4.3

(a) Assuming $E < 0$ the states can be written:

$$\phi_1(x) = A\exp(\kappa x) \tag{81}$$

$$\phi_2(x) = B\cos(kx) + C\sin(kx) \tag{82}$$

$$\phi_3(x) = D\exp(-\kappa x). \tag{83}$$

Find expressions for κ and k as functions of energy.

(b) Why must κ be the same in regions 1 and 3?

(c) Why must the wavefunction be symmetric or antisymmetric? State the constraints on the coefficients in each case.

(d) Use the boundary conditions to find constraints on k and κ

(e) Show how you could go about solving these equations graphically (without actually doing so).

(f) Prove there must always be at least one bound state in the well regardless of how small V_0 is.

4.4.4

Why can the unbound states not be normalized?

4.4.5

Sketch the form of an unbound state, paying attention to the relative amplitudes of the wavefunction in the various regions.