5 Finite-dimensional Hilbert spaces

Bring solutions along to the class at 10am on 2/11/20.

Questions marked # are beyond the syllabus. We may discuss them in the class if there is time. Attempt all questions. Feel free to discuss with friends. Don't worry if you cannot answer a question fully as you will discuss the solutions in the class.

In problem set P0 we saw a convenient notation for complex vectors, $|v\rangle$. This problem set develops some useful properties of complex vectors.

5.0 Videos

Please watch this week's videos: V5.1, V5.2, V5.3a, V5.3b, V5.3c, V5.4

All videos are available on the youtube channel Introductory Quantum Mechanics (link available at felixflicker.com/teaching, or on Learning Central).

5.1 Hermitian conjugate

5.1.1

Define a complex N-dimensional vector

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \end{pmatrix}. \tag{84}$$

Write expressions for

(a) the complex conjugate $(|v\rangle)^*$

(b) the transpose $(|v\rangle)^T$.

What are the matrix dimensions of each object?

5.1.2

Show that $\left((|v\rangle)^T\right)^* = \left((|v\rangle)^*\right)^T$.

N.B The relevance is that we can therefore write $(|v\rangle)^{T*}$ without ambiguity. This is called the Hermitian conjugate $(|v\rangle)^\dagger$:

$$(|v\rangle)^\dagger \triangleq (|v\rangle)^{T*}. \tag{85}$$

By convention we write this

$$(|v\rangle)^\dagger \triangleq \langle v|. \tag{86}$$

5.2 Inner product

5.2.1

Show that the inner product $\langle u||v\rangle$, which we conventionally write $\langle u|v\rangle$, is a complex scalar.

5.2.2

Show that

$$(\langle u|v\rangle)^* = \langle v|u\rangle. \tag{87}$$

5.2.3

Show that the norm of $|v\rangle$, *i.e.* its length, denoted $\||v\rangle\|$, is given by

$$\||v\rangle\| = \sqrt{\langle v|v\rangle}. \tag{88}$$

5.2.4

Let

$$|v\rangle = \alpha|u\rangle + \beta|w\rangle \tag{89}$$

with complex α and β. Assuming $|u\rangle$ and $|w\rangle$ are orthogonal and normalised, find a condition on α and β for $|v\rangle$ to also be normalised.

5.3 Matrices acting on vectors

Assume A is an $N \times N$ complex matrix, and $|u\rangle$ and $|v\rangle$ are N-dimensional complex vectors. State the dimensions of the following objects:

5.3.1 $A|v\rangle$

5.3.2 $\langle v|A$

5.3.3 $\langle u|A|v\rangle$.

5.4 Outer product

In general we denote the outer product (also called the tensor product) between two N-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ as

$$\boldsymbol{u} \otimes \boldsymbol{v} \triangleq \boldsymbol{u}\boldsymbol{v}^\dagger.$$

Element-wise the statement is that

$$[\boldsymbol{u} \otimes \boldsymbol{v}]_{ij} = u_i v_j^*$$

(*i.e.* multiply element-by-element).

In general the vectors need not be the same length, but we will assume they are.

(a) Thinking again of N-dimensional vectors as $N \times 1$ matrices, explain why the outer product is an $N \times N$ matrix.

(b) Show that $|v\rangle\langle u|$ is an outer product.

5.5 Resolution of the identity

Assume $|e_i\rangle$ ($i \in [1, N]$) form a complete orthonormal basis.

(a) What is $\langle e_i|e_j\rangle$?

(b) For an arbitrary vector $|v\rangle$ with elements v_i explain why

$$|v\rangle = \sum_{i=1}^{N} v_i |e_i\rangle \tag{90}$$

where

$$v_i = \langle e_i|v\rangle. \tag{91}$$

(c) Two matrices A and B are equivalent iff

$$\langle u|A|v\rangle = \langle u|B|v\rangle \tag{92}$$

for all $|u\rangle$, $|v\rangle$. Using the results of (b) prove the *resolution of the identity*

$$\mathbb{I} = \sum_{i=1}^{N} |e_i\rangle\langle e_i| \tag{93}$$

where $\mathbb{I}$ is the $N \times N$ identity matrix.

5.6 Hermitian matrices

A Hermitian matrix is one which is equal to its Hermitian conjugate:

$$A^\dagger = A \tag{94}$$

i.e.

$$A^{T*} = A \tag{95}$$

or

$$A_{ij}^* = A_{ji}. \tag{96}$$

(a) Explain why this requires A to be square. We will assume this from now on.

(b) Recalling that

$$(AB)^T = B^T A^T \tag{97}$$

explain why

$$(A|v\rangle)^\dagger = \langle v|A^\dagger. \tag{98}$$

(c) The eigenvectors $|v_n\rangle$ and eigenvalues λ_n of matrix A are defined by the equation:

$$A|v_n\rangle = \lambda_n|v_n\rangle. \tag{99}$$

Show that

$$\langle v_n|A^\dagger = \lambda_n^*\langle v_n|. \tag{100}$$

(d) By considering the object

$$\langle v_n|\left(A - A^\dagger\right)|v_n\rangle \tag{101}$$

prove that Hermitian matrices have real eigenvalues.

(e) Considering instead the object

$$\langle v_m|\left(A - A^\dagger\right)|v_n\rangle \tag{102}$$

prove the the eigenvectors of a Hermitian matrix are orthogonal to one another, provided no two eigenvalues are the same (the matrix is non-degenerate).

(f) Assuming the eigenvalues of an $N \times N$ Hermitian matrix are non-degenerate, explain why the normalised eigenvectors must form a basis for the N-dimensional linear vector space.

5.7 Matrices and eigenvalues

5.7.1

Using the resolution of the identity, show that any Hermitian matrix M can be written in the following form:

$$M = \sum_{i=1}^{N} \lambda_i |i\rangle\langle i| \tag{103}$$

where

$$M|i\rangle = \lambda_i|i\rangle. \tag{104}$$

5.7.2

Show this explicitly for each of the three Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{105}$$

5.8 Functions of matrices

A function of a matrix can be defined in terms of its Taylor series.

5.8.1

By comparing their Taylor series, show that

$$\exp(iAx) = \cos(Ax) + i\sin(Ax). \tag{106}$$

5.8.2

Using the result that

$$\sigma_x^2 = \mathbb{I} \tag{107}$$

where σ_x is a Pauli matrix and $\mathbb{I}$ is the 2×2 identity matrix, show that

$$\exp(i\sigma_x x) = \cos(x)\,\mathbb{I} + i\sin(x)\,\sigma_x. \tag{108}$$

5.8.3

What are the eigenvectors of $\exp(i\sigma_x x)$? What are the corresponding eigenvalues?

5.8.4

Show that for

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{109}$$

we have

$$\exp(M) = \mathbb{I} + M + \sum_{n=2}^{\infty} \frac{1}{n!} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{110}$$

where F_n is the n^{th} Fibonacci number.

5.9 Spin-1/2

5.9.1

The spin operators are defined to be

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i \tag{111}$$

with $i = x, y, z$. Show that each of the three spin operators has eigenvalues $\pm\hbar/2$ and find the corresponding normalised eigenvectors.

5.9.2

Denote the eigenvector of $\hat{S}_i$ with eigenvalue $+\hbar/2$ with the symbol $|\uparrow_i\rangle$, and that with eigenvalue $-\hbar/2$ with the symbol $|\downarrow_i\rangle$:

$$\hat{S}_i|\uparrow_i\rangle = \frac{\hbar}{2}|\uparrow_i\rangle$$
$$\hat{S}_i|\downarrow_i\rangle = -\frac{\hbar}{2}|\downarrow_i\rangle.$$

Using the result of equation 103 explain why each spin operator can be written in the form

$$\hat{S}_i = \frac{\hbar}{2}\left(|\uparrow_i\rangle\langle\uparrow_i| - |\downarrow_i\rangle\langle\downarrow_i|\right). \tag{112}$$

5.9.3

Show that we can write all operators in the spin-z basis as follows:

$$\hat{S}_x = \frac{\hbar}{2}\left(|\uparrow_z\rangle\langle\downarrow_z| + |\downarrow_z\rangle\langle\uparrow_z|\right) \tag{113}$$
$$\hat{S}_y = \frac{\hbar}{2}\left(i|\downarrow_z\rangle\langle\uparrow_z| - i|\uparrow_z\rangle\langle\downarrow_z|\right) \tag{114}$$
$$\hat{S}_z = \frac{\hbar}{2}\left(|\uparrow_z\rangle\langle\uparrow_z| - |\downarrow_z\rangle\langle\downarrow_z|\right). \tag{115}$$

If you see spins written without the direction designated it is conventional to define them along z in this way.

5.9.4

Use the forms just derived to show that

$$\left[\hat{S}_i, \hat{S}_j\right] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}\hat{S}_k \tag{116}$$

where the directions x, y, z are numbered 1, 2, and 3 respectively, and ϵ_{ijk} is the Levi-Civita symbol defined by

$$\epsilon_{ijk} = \begin{cases} 0, \text{ any of } i, j, k \text{ equal} \\ 1, \; ijk = 123 \text{ or cyclic permutations} \\ -1, \; ijk = 321 \text{ or cyclic permutations.} \end{cases} \tag{117}$$