7 Quantum mechanics

Bring solutions along to the class at 10am on 16/11/20.

Questions marked # are beyond the syllabus. We may discuss them in the class if there is time. Attempt all questions. Feel free to discuss with friends. Don't worry if you cannot answer a question fully as you will discuss the solutions in the class.

7.0 Videos

Please watch this week's videos: V7.1, V7.2, V7.3, V7.4, V7.5.
All videos are available on the youtube channel Introductory Quantum Mechanics (link available at felixflicker.com/teaching, or on Learning Central).

7.1 Infinite-dimensional Hilbert spaces

7.1.1

State the axioms of linear vector spaces (recall PS0), and show that they are satisfied by complex functions $f(x)$. We are therefore justified in using vector notation to decribe functions: $|f\rangle$, or, in the position basis specifically, $\langle x|f\rangle = f(x)$.

7.1.2

State the axioms of an 'inner product space'. Working in the position basis $f(x) = \langle x|f\rangle$ show that the axioms are satisfied by complex functions if the inner product is defined as

$$\langle f|g\rangle \triangleq \int_{-\infty}^{\infty} f(x)^* g(x)\,\mathrm{d}x. \tag{150}$$

7.1.3

Explain why the norm of any quantum state $|\psi\rangle$ must be one. State this restriction in terms of the wavefunction $\langle x|\psi\rangle = \psi(x)$. This last property, square-integrability, defines the space of wavefunctions to be an infinite-dimensional Hilbert space.

7.2 Fourier transforms

In general the Fourier transform (FT) and inverse transform (FT^{-1}) of a pair of functions in variables ξ and t

$$\tilde{f}(\xi) = \mathrm{FT}[f(t)] \tag{151}$$
$$f(t) = \mathrm{FT}^{-1}\left[\tilde{f}(\xi)\right] \tag{152}$$

are defined as:

$$\tilde{f}(\xi) = \int_{-\infty}^{\infty} f(t) \exp(-2\pi i \xi t)\, \mathrm{d}t \tag{153}$$
$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\xi) \exp(2\pi i \xi t)\, \mathrm{d}\xi \tag{154}$$

[t need not be time; ξ and t are just any pair of Fourier-conjugate variables]. In quantum mechanics it is convenient to rescale the variables to

$$2\pi\xi t \to px/\hbar \tag{155}$$

so that we are working with plane waves. One way to achieve this is to define

$$\xi \triangleq p/\sqrt{2\pi\hbar} \tag{156}$$
$$t \triangleq x/\sqrt{2\pi\hbar}. \tag{157}$$

There are other choices, akin to the usual ambiguity in the placement of the 2π factors in Fourier transforms.

7.2.1

For a suitable definition of $\phi(x)$ show that the definitions of Eqns. 156 and 157 lead to

$$\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(x) \exp(-ipx/\hbar)\, \mathrm{d}x$$
$$\phi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{\phi}(p) \exp(ipx/\hbar)\, \mathrm{d}p.$$

7.2.2

(a) We can resolve the identity $\hat{\mathbb{I}}$ into either the position basis:

$$\hat{\mathbb{I}} = \int_{-\infty}^{\infty} |x\rangle\langle x| \mathrm{d}x \tag{158}$$

or momentum basis:

$$\hat{\mathbb{I}} = \int_{-\infty}^{\infty} |p\rangle\langle p| \mathrm{d}p. \tag{159}$$

Define the ket $|\phi\rangle$ such that its projection into the position basis is $\phi(x)$, *i.e.*

$$\langle x|\phi\rangle \triangleq \phi(x)\,. \tag{160}$$

For consistency we would like the projection of $|\phi\rangle$ into the momentum basis to be the Fourier transform of this:

$$\langle p|\phi\rangle = \tilde{\phi}(p) \,. \tag{161}$$

That is, we would like

$$\begin{aligned} \tilde{\phi}(p) &= \mathrm{FT}\left[\phi(x)\right] & (162) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(-ipx/\hbar\right)\phi(x)\,\mathrm{d}x. & (163) \end{aligned}$$

By inserting a resolution of the identity in the position basis (Eq. 158) into Eq. 161 show that we require

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-ipx/\hbar\right). \tag{164}$$

(b) We expect $\phi(x)$ to be the inverse Fourier transform (FT^{-1}) of $\tilde{\phi}(p)$, *i.e.*

$$\begin{aligned} \phi(x) &= \mathrm{FT}^{-1}\left[\tilde{\phi}(p)\right] & (165) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(ipx/\hbar\right)\tilde{\phi}(p)\,\mathrm{d}p. & (166) \end{aligned}$$

Show that this implies

$$\left(\langle p|x\rangle\right)^* = \langle x|p\rangle. \tag{167}$$

Why is a relation of this form desirable?

(c) Since $|p\rangle$ is a momentum eigenstate, we expect $\langle x|p\rangle$ to be the wavefunction of a momentum eigenstate written in the position basis, *i.e.*

$$\hat{p}\left(\langle x|p\rangle\right) = -i\hbar\partial_x\left(\langle x|p\rangle\right) = p\left(\langle x|p\rangle\right). \tag{168}$$

Check that this is indeed the case.

7.2.3

(a) The states $\{|x\rangle\}$ are orthonormal, *i.e.*

$$\langle x|x'\rangle = \delta\left(x - x'\right) \tag{169}$$

where the Dirac delta function is defined by

$$\int_{-\infty}^{\infty} \delta\left(x-x'\right) f\left(x'\right) \mathrm{d}x' = f\left(x\right). \tag{170}$$

Show that this implies the relation

$$2\pi\delta\left(x\right) = \frac{1}{\hbar}\int_{-\infty}^{\infty} \exp\left(ixp/\hbar\right) \mathrm{d}p. \tag{171}$$

(b) The states $\{|p\rangle\}$ are also orthonormal, *i.e.*

$$\langle p|p'\rangle = \delta\left(p-p'\right). \tag{172}$$

Show that this gives

$$2\pi\delta\left(p\right) = \frac{1}{\hbar}\int_{-\infty}^{\infty} \exp\left(ixp/\hbar\right) \mathrm{d}x.$$

7.2.4

Using the forms

$$\mathrm{FT}\left[\phi\left(x\right)\right] = \int_{-\infty}^{\infty} \langle p|x\rangle \phi\left(x\right) \mathrm{d}x \tag{173}$$

$$\mathrm{FT}^{-1}\left[\tilde{\phi}\left(p\right)\right] = \int_{-\infty}^{\infty} \langle x|p\rangle \tilde{\phi}\left(p\right) \mathrm{d}p \tag{174}$$

and Eqs. 169 and 172 show that

$$\tilde{\phi}\left(p\right) = \mathrm{FT}\left[\mathrm{FT}^{-1}\left[\tilde{\phi}\left(p\right)\right]\right] \tag{175}$$

and

$$\phi\left(x\right) = \mathrm{FT}^{-1}\left[\mathrm{FT}\left[\phi\left(x\right)\right]\right]. \tag{176}$$

7.3 Justifying the differential operators

7.3.1

A function of an operator $\phi\left(\hat{x}\right)$ can be defined by its Taylor series. Defining an arbitrary function $f\left(x\right)$ such that

$$\hat{x} f\left(x\right) = x f\left(x\right) \tag{177}$$

show that

$$\left[\phi\left(\hat{x}\right), \hat{p}\right] f\left(x\right) = i\hbar \frac{\partial \phi\left(x\right)}{\partial x} f\left(x\right). \tag{178}$$

7.3.2

By expanding the commutator (or otherwise) explain why it is natural to assign $\hat{p} f(x) = -i\hbar \partial_x f(x)$.

7.4 Hermiticity of differential operators

7.4.1

State the condition for a differential operator $\hat{A}$ (written in the position basis) to be Hermitian.

7.4.2

State whether each of the following operators, written in the position basis, is Hermitian. If it is not, state its Hermitian conjugate.

(a) x

(b) ∂_x

(c) $-i\hbar\partial_x$

(d) ∂_x^2

(e) $-i\hbar(x\partial_y - y\partial_x)$

(f) ∇