8 The quantum harmonic oscillator

Bring solutions along to the class at 10am on 23/11/20.

Questions marked # are beyond the syllabus. We may discuss them in the class if there is time. Attempt all questions. Feel free to discuss with friends. Don't worry if you cannot answer a question fully as you will discuss the solutions in the class.

8.0 Videos

Please watch this week's videos: V8.1, V8.2, V8.3, V8.4.

All videos are available on the youtube channel Introductory Quantum Mechanics (link available at felixflicker.com/teaching, or on Learning Central).

8.1 Solution using Hermite polynomials

The Hamiltonian for the quantum harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{180}$$

8.1.1

Working in the position basis, define

$$x = \alpha y \tag{181}$$

for suitable α to show that the TISE can be written

$$-\frac{1}{2}\frac{\mathrm{d}^2\phi_n(y)}{\mathrm{d}y^2} + \frac{1}{2}y^2\phi_n(y) = \epsilon_n\phi_n(y) \tag{182}$$

and state an expression for ϵ_n.

8.1.2

Using the ansatz

$$\phi_n(y) = H_n(y)\exp\left(-y^2/2\right) \tag{183}$$

show that Eq. 182 can be rewritten as Hermite's equation

$$H_n(y)'' - 2yH_n(y)' + (2\epsilon - 1)H_n(y) = 0. \tag{184}$$

8.1.3

The solutions are 'Hermite polynomials', defined by:

$$H_0 = 1 \tag{185}$$
$$H_{n\geq 1}(y) = (-1)^n \exp\left(y^2\right) \frac{\mathrm{d}^n}{\mathrm{d}y^n} \exp\left(-y^2\right). \tag{186}$$

Find the first four Hermite polynomials explicitly, and sketch them. Therefore, sketch the first four energy eigenstates of the harmonic oscillator $\phi_n(x)$.

8.2 Ladder operators

8.2.1

Continuing with the rescaled variable from the previous question, the Hamiltonian in the position basis is

$$\hat{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{2}y^2. \tag{187}$$

Show this can be rewritten as

$$\hat{H} = \hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{\mathbb{I}} \tag{188}$$

where the lowering operator is

$$\hat{a} = \frac{1}{\sqrt{2}}\left(y + \frac{\mathrm{d}}{\mathrm{d}y}\right) \tag{189}$$

and the raising operator is

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(y - \frac{\mathrm{d}}{\mathrm{d}y}\right) \tag{190}$$

(collectively they are known as ladder operators).

8.2.2

Show, using the definition of the Hermitian conjugate for differential operators, that $\hat{a}^\dagger$ is the Hermitian conjugate of $\hat{a}$.

8.2.3

Show that

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = \hat{\mathbb{I}}. \tag{191}$$

8.2.4

Show that

$$\left[\hat{a}^{\dagger}, \hat{H}\right] = -\hat{a}^{\dagger}. \tag{192}$$

8.2.5

By acting on both sides of the time-independent Schrödinger equation

$$\hat{H}\phi_n(y) = \epsilon_n \phi_n(y) \tag{193}$$

with $\hat{a}^{\dagger}$, show that

$$\hat{H}\left(\hat{a}^{\dagger}\phi_n(y)\right) = (\epsilon_n + 1)\left(\hat{a}^{\dagger}\phi_n(y)\right).$$

8.2.6

Explain how the previous result proves that the harmonic oscillator has an infinite ladder of energy eigenstates evenly-spaced in energy.

8.2.7

Explain why it must be the case that

$$\int_{-\infty}^{\infty} \left|\hat{a}\phi_n(y)\right|^2 \mathrm{d}y \geq 0. \tag{194}$$

Hence argue that there is a lowest rung to the ladder of energies.

8.2.8

The eigenstate on the lowest rung must obey this condition:

$$\hat{a}\phi_0(y) = 0. \tag{195}$$

Use this equation to solve for the normalised eigenstate $\phi_0(y)$ and the corresponding energy eigenvalue ϵ_0.

8.2.9

Find the normalised first excited state and the corresponding energy eigenvalue.

8.2.10

Explain why

$$\hat{a}^\dagger \hat{a} \phi_n \propto \phi_n. \tag{196}$$

By multiplying both sides by ϕ_n^* and integrating over all y, show that the constant of proportionality is n.

8.2.11

Show that

$$\hat{a}\phi_n = \sqrt{n}\phi_{n-1}. \tag{197}$$

8.2.12

Show that

$$\hat{a}^\dagger \phi_n = \sqrt{n+1}\phi_{n+1}. \tag{198}$$

8.2.13

Hence explain why we can write any normalised eigenstate of the harmonic oscillator as

$$\phi_n(y) = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n \phi_0(y). \tag{199}$$

8.3 Ladder operators redone

In this question we will redo question 8.2 in a basis-independent manner, using just the canonical commutation relations and Dirac notation. We will not rescale the variables, partly because this is less necessary and partly because it's good to be familiar with different ways of approaching the problem. Check you understand how each result relates to the corresponding result in 8.2.

8.3.1

The Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{200}$$

Show this can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\mathbb{I}}\right) \tag{201}$$

where the lowering operator is

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \tag{202}$$

and the raising operator is

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) \tag{203}$$

(collectively they are known as ladder operators).

8.3.2

Show that $\hat{a}^\dagger$ is the Hermitian conjugate of $\hat{a}$.

8.3.3

Show that

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{\mathbb{I}}. \tag{204}$$

8.3.4

Show that

$$\left[\hat{a}^\dagger, \hat{H}\right] = -\hbar\omega\hat{a}^\dagger. \tag{205}$$

8.3.5

By acting on both sides of the time-independent Schrödinger equation

$$\hat{H}|n\rangle = E_n|n\rangle \tag{206}$$

with $\hat{a}^\dagger$, show that

$$\hat{H}\left(\hat{a}^\dagger|n\rangle\right) = \left(E_n + \hbar\omega\right)\left(\hat{a}^\dagger|n\rangle\right).$$

8.3.6

Explain how the previous result proves that the harmonic oscillator has an infinite ladder of energy eigenstates evenly-spaced in energy.

8.3.7

Explain why it must be the case that

$$\| \hat{a}|n\rangle \|^2 \geq 0. \tag{207}$$

Hence argue that there is a lowest rung to the ladder of energies.

8.3.8

The eigenstate on the lowest rung, $|0\rangle$ (just a state labelled 0, not the number zero!), must obey this condition:

$$\hat{a}|0\rangle = 0. \tag{208}$$

Use this equation to solve for the normalised eigenstate $\langle x|0\rangle = \varphi_0(x)$ and the corresponding energy eigenvalue E_0. Here we do need to work in the position basis.

8.3.9

Find the normalised first excited state (in the position basis) and the corresponding energy eigenvalue.

8.3.10

Explain why

$$\hat{a}^\dagger\hat{a}|n\rangle \propto |n\rangle. \tag{209}$$

Acting from the left with $\langle n|$ show that the constant of proportionality is n.

8.3.11

Show that

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle. \tag{210}$$

8.3.12

Show that

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{211}$$

8.3.13

Hence explain why we can write any normalised eigenstate of the harmonic oscillator as

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle. \tag{212}$$

8.4 Second quantisation

Explain what is meant by second quantisation.