9 The Schrödinger equation in three dimensions

Bring solutions along to the class at 10am on 30/11/20.

Questions marked # are beyond the syllabus. We may discuss them in the class if there is time. Attempt all questions. Feel free to discuss with friends. Don't worry if you cannot answer a question fully as you will discuss the solutions in the class.

9.0 Videos

Please watch this week's videos: V9.1, V9.2, V9.3, V9.4.
All videos are available on the youtube channel Introductory Quantum Mechanics (link available at felixflicker.com/teaching, or on Learning Central).

9.1

The three-dimensional infinite-potential (cubic) well is defined by the potential

$$V(\boldsymbol{r}) = \begin{cases} 0, & 0 \leq r_i \leq L \\ \infty, & \text{otherwise} \end{cases} \tag{213}$$

where $\boldsymbol{r} = (x, y, z)$ and r_i is element i of $\boldsymbol{r}$.

9.1.1

Sketch the potential.

9.1.2

Write down the time-independent Schrödinger equation for this potential.

9.1.3

Find the energy eigenvalues and normalised energy eigenstates.

9.1.4

The degeneracy of an energy eigenstate is the number of other energy eigenstates with the same energy eigenvalue. Find the degeneracy of the lowest five energy levels of the 3D infinite potential well.

9.1.5

Imagine the well's shape is cuboidal instead of cubic. What would be the degeneracy of the lowest five energy levels in this case?

9.2 Angular momentum operators

9.2.1

State the commutation relation between any two angular momentum operators. Show that

$$\left[\hat{L}^2, \hat{L}_z\right] = 0. \tag{214}$$

9.2.2

Denote the eigenstates of $\hat{L}_z$ and $\hat{L}^2$ as follows:

$$\hat{L}^2|l\rangle = \hbar^2 l\,(l+1)\,|l\rangle \tag{215}$$

$$\hat{L}_z|m\rangle = \hbar m|m\rangle \tag{216}$$

where l and m are integers. Why is it reasonable to write a state such as $|l, m\rangle$ labelled by both eigenvalues?

9.2.3

Defining

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y \tag{217}$$

show that

$$\left[\hat{L}_\pm, \hat{L}_z\right] = \mp\hbar\hat{L}_\pm. \tag{218}$$

9.2.4

Using the reasoning you developed with the quantum harmonic (energy) ladder operators, explain why the previous result defines $\hat{L}_\pm$ to be angular momentum raising and lowering operators.

9.3

Consider the two-dimensional quantum harmonic oscillator defined by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x}^2 + \hat{y}^2\right). \tag{219}$$

9.3.1

Explain why you might correctly expect that

$$[\hat{p}_x, \hat{p}_y] = [\hat{p}_x, \hat{y}] = [\hat{p}_y, \hat{x}] = [\hat{x}, \hat{y}] = 0.$$

9.3.2

Show that the Hamiltonian can be rewritten

$$\hat{H} = \hbar\omega\left(\hat{n}_x + \hat{n}_y + \hat{\mathbb{I}}\right) \tag{220}$$

where

$$\hat{n}_x = \hat{a}_x^\dagger \hat{a}_x \tag{221}$$

and

$$\hat{a}_x = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right) \tag{222}$$

and similarly for y.

9.3.3

Explain why it is reasonable to label the energy eigenstates

$$\hat{H}|n_x, n_y\rangle = E_{n_x,n_y}|n_x, n_y\rangle. \tag{223}$$

State an expression for $|n_x, n_y\rangle$ in terms of the ground state $|0, 0\rangle$.

9.3.4

Defining the operator

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{224}$$

show that

$$\hat{L}_z = i\hbar\left(\hat{a}_y^\dagger \hat{a}_x - \hat{a}_x^\dagger \hat{a}_y\right). \tag{225}$$

Hence show that $\hat{L}_z$ commutes with neither $\hat{n}_x$ nor $\hat{n}_y$ individually, but does commute with $\hat{H}$. What does this tell us about the possible quantum numbers? Hence suggest an alternate labelling of the energy eigenstates.

9.4

[#: you won't need to reproduce the derivation in an exam, but you should know where it comes from.] Starting from the definition of classical angular momentum in cartesian co-ordinates, derive the quantum operator $\hat{L}_z$ in spherical polar co-ordinates.