A Practical Intoduction to QFT: Questions

These questions are completely optional. Some questions do not require anything to be written down. In all cases collaboration and discussion is thoroughly encouraged. Starred (*) questions are either hard, very hard, or require use of index manipulation which is not covered in this course.

1 Actions and Lagrangians

1.1 Natural Units

[Copied from my first year notes on dimensional analysis] Choosing our one dimension to be energy $\mathbb{E}$, we can say for example $[E] = [m] = 1$, $[x] = [t] = -1$, where the number indicates the power of $\mathbb{E}$ (no ambiguity since we only have one dimension).
(a) Calculate the dimensions of the following fields given that $[S] = 0$ and $[\partial] = 1$.

- $S = \int \mathrm{d}^4x \left\{-\frac{1}{2}(\partial\varphi)^2\right\}$
- $S = \int \mathrm{d}^4x \left\{-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{4!}\varphi^4\right\}$ ($g$and $m$are not fields)
- $S = \int \mathrm{d}^3x \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho$ ($[\epsilon]$=0)
- $S = \int \mathrm{d}^3x \epsilon_{\mu\nu\rho}\left(A_a^\mu \partial^\nu A_a^\rho + g f^{abc} A_a^\mu A_b^\nu A_c^\rho\right)$ ($[f] = 0$)
- $S = \int \mathrm{d}^4x \psi^\dagger (i\partial_t - H)\psi$
- $S = \int \mathrm{d}^3x \psi^\dagger (i\partial_t - H)\psi$
- $S = \int \mathrm{d}^4x \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$
- $S = \int \mathrm{d}^4x \left\{-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\mu - \partial^\nu A^\mu) + \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - gA_\mu\overline{\psi}\gamma^\mu\psi\right\}$

If the coupling constant g has dimension $[g] > 0$ the theory is said to be 'relevant'. If $[g] < 0$ it is irrelevant, and if $[g] = 0$ it is marginal.
(b) Classify the three cases above.

1.2 Functional Derivatives

In the lectures we found that the shortest distance between two points, with no additional weighting, is a straight line. This is the 'geodesic' light would follow in Euclidean space. An additional weighting would come in for example if we were to consider light moving in a medium, in which case the weight is the refractive index.
(a) If the measure of length is

$$L[f] = \int_{x_0}^{x_1} \mathrm{d}x n(x) \sqrt{1 + f'(x)^2}$$

show by varying the functional that minimal paths now obey

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{n(x) f'(x)}{\sqrt{1 + f'(x)^2}}\right).$$

(b) By considering an infinitesimal part of the line $f(x)$ show that

$$\frac{f'(x)}{\sqrt{1+f'(x)^2}} = \sin(\vartheta)$$

where ϑ is the angle between the tangent to the curve and the x-axis.
(c) By considering a refractive index of the form

$$n(x) = \begin{cases} n_-, & x < 0 \\ n_+, & x > 0 \end{cases}$$

deduce Snell's law.

1.3 Euler-Lagrange Equations

A generic action functional of a real scalar field φ can be written

$$S[\varphi] = \int \mathrm{d}^4 x \mathscr{L}(\varphi, \partial_\mu \varphi).$$

(a) Show (using the $\varphi(x) + \lambda\epsilon(x)$ method) that if φ extremises the action the Lagrange density obeys the Euler-Lagrange equation:

$$0 = \frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \right).$$

(b) Find the Euler-Lagrange equations for the actions listed in the first lecture, either using the result of (a) or by varying explicitly again. If you're unhappy with Einstein summation notation skip Maxwell and Chern-Simons:

$$\begin{aligned}
S_{\text{Klein-Gordon}}[\varphi] &= \int \mathrm{d}^4 x \left(-\frac{1}{2}\varphi\Box\varphi - \frac{1}{2}m^2\varphi^2 + j\varphi \right) \\
S_{\text{Maxwell}}[A_\mu] &= \int \mathrm{d}^4 x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu \right), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\
S_{\text{Schrödinger}}[\psi] &= \int \mathrm{d}^3\mathbf{x}\mathrm{d}t \left(\psi^\dagger (i\partial_t - H)\psi + j\psi^\dagger + j^\dagger \psi \right) \\
S_{\text{Chern-Simons}}[A_\mu] &= \int \mathrm{d}^3 x \left(\epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho + j_\mu A^\mu \right) \\
S_{\text{Klein-Gordon}}[\tilde{\varphi}] &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left(\frac{1}{2}\tilde{\varphi}\left(p^2 - m^2\right)\tilde{\varphi} + \tilde{j}\tilde{\varphi} \right).
\end{aligned}$$

(c) The sine-Gordon action is

$$S_{\text{sine-Gordon}}[\varphi] = \int \mathrm{d}^4 x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \cos(\varphi) \right].$$

Find the Euler Lagrange equation and hence explain the name.

1.4 Maxwell-Chern Simons Theory*

An example of a tricky quadratic theory is defined by the Maxwell-Chern Simons Lagrangian density

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{im}{2}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 + j_\mu A^\mu$$

where $\epsilon_{\mu\nu\rho}$ is the antisymmetric (Levi Civita) tensor. The Chern Simons term only works in $2+1$D, and is know to provide a low energy effective field theory for anyons (worth a look-up on wikipedia!). A gauge fix into the Lorenz gauge is carried out by the third term upon sending $\alpha \to 0$.
(i) Show that the Euler Lagrange equation, with a δ-function current, is given in reciprocal space by

$$\left(-k^2\eta_{\alpha\mu} + \left(1 - \frac{1}{\alpha}\right)k_\alpha k_\mu + m\epsilon_{\mu\beta\alpha}k^\beta \right)\tilde{G}^{\mu\nu}(k) = \delta_\alpha^\nu$$

(you will find it helpful to follow the working for the Maxwell theory in the notes).
(ii) Show by substitution that this is a valid solution:

$$\tilde{G}^{\mu\nu}(k) = \frac{-1}{k^2+m^2}\left(\eta^{\mu\nu} - \left(1-\left(k^2+m^2\right)\frac{\alpha}{k^2}\right)\frac{k^\mu k^\nu}{k^2} + \frac{m}{k^2}\epsilon^{\mu\nu\gamma}k_\gamma\right)$$

(here it is helpful to note that $k^\mu k^\nu$ is symmetric, while $\epsilon^{\mu\nu\rho}$ is antisymmetric. I found it easiest to write out the 9 terms separately then look to see which combine naturally, and which cancel).

2 Partition Functions

2.1 0+0D QFT

2.1.1 The first trick in the book

QFTs can be defined in arbitrary numbers of dimensions. The simplest is to have a field which takes values in a 0D space and 0D time (*i.e.* no space or time dependence). Then the field ϕ is just a variable rather than a function (let's say it's real). All the usual calculational methods follow through.
Let

$$\frac{\mathscr{Z}_j}{\mathscr{Z}_0} \triangleq \frac{\int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(i\left(-\frac{a}{2}\phi^2 + j\phi\right)\right)}{\int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{a}{2}i\phi^2\right)}.$$

By changing variables to $\phi' = \phi + \Phi$, with Φ fixed and chosen to cancel the linear term of ϕ', show that

$$\frac{\mathscr{Z}_j}{\mathscr{Z}_0} = \exp\left(ij^2/2a\right).$$

2.1.2 Functional averages

We define the functional average of an operator $\mathcal{O}[\phi]$ to be

$$\langle\mathcal{O}[\phi]\rangle \triangleq \frac{\int_{-\infty}^{\infty} \mathrm{d}\phi \mathcal{O}[\phi] \exp\left(i\left(-\frac{a}{2}\phi^2 + j\phi\right)\right)}{\int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{a}{2}i\phi^2\right)}.$$

Show that

$$\langle\phi\phi\rangle = \left(-i\frac{\partial}{\partial j}\right)^2 \frac{\mathscr{Z}_j}{\mathscr{Z}_0}\bigg|_{j=0}.$$

By taking derivatives of both forms of $\mathscr{Z}_j/\mathscr{Z}_0$ separately (then setting $j=0$) show that

$$\begin{aligned} A: \quad & \langle\phi\rangle &=& \; 0 \\ B: \quad & \langle\phi\phi\rangle &=& \; -i/a \\ C: \quad & \left\langle\phi^{2n}\right\rangle &=& \; const. \times \Pi_{i=1}^{n}\langle\phi\phi\rangle \\ D: \quad & \left\langle\phi^{2n+1}\right\rangle &=& \; 0. \end{aligned}$$

From top to bottom the results are: (A) zero vacuum expectation value for the field (a sensible choice in most QFTs except, notably, the Higgs field); (B) the Green's function for this trivial theory; (C,D) Wick's theorem.

2.1.3 Diagrammatics

Wick's theorem really follows from the simple realisation that the only object in the free theory (about which you are always expanding) is the 2-point correlator (propagator). Diagrammatically a propagator is represented by a line segment. By drawing a collection of dots and connecting them with line segments, such that each line segment terminates on two points, convince yourself of conclusions (C) and (D) in the last question. What does (A) look like diagrammatically?
Imagine if we instead had a weird (0+0D) field theory with a cubic action

$$S[x] = \frac{a}{3}x^3.$$

Calculate algebraically (*i.e.* by taking derivatives as before) the equivalents to (A)-(D). The fundamental Green's function is now the 3-point correlator, which you could represent with a triangle. Draw a collection of points and show your conclusions diagrammatically.
Finally, do the same for a quartic action

$$S[x] = \frac{a}{4}x^4.$$

There are two obvious reasons why the cubic action is not acceptable as a free field theory. One of these reasons applies to the quartic theory as well. What are they?[1]

2.2 0+1D QFT

You sometimes hear people describe quantum mechanics as a 0+1D quantum field theory[2].
(a) By considering the generic form of the Lagrangian (not Lagrange density), convince a friend why the quantum mechanical amplitude to propagate from point $q(0)$ to point $q(t)$ is given by

$$G = \int \mathscr{D}q \exp\left(i \int \mathrm{d}t \left[\frac{m}{2}\dot{q}^2 - V(q)\right]\right).$$

(b) Using the result for infinite dimensional gaussian integrals find the Green's function explicitly for the case of the simple harmonic oscillator, $V(q) = \frac{1}{2}m\omega^2 q^2$, in terms of the determinant of an operator.
(c) Discuss with another friend in what sense QM = 0+1D QFT.

2.3 3+1D QFT

It's a simple generalization to consider higher dimensional, say 3+1D, QFTs. Reproduce the results of Sections 2.1.1 and 2.1.2 using the real scalar field defined on a 3+1D spacetime, starting from

$$\frac{\mathscr{Z}_j}{\mathscr{Z}_0} \triangleq \frac{\int \mathscr{D}\phi \exp\left(i \int \mathrm{d}^4x \left(-\frac{a}{2}\phi(x)^2 + j(x)\phi(x)\right)\right)}{\int \mathscr{D}\phi \exp\left(-\frac{a}{2} i \int \mathrm{d}^4x \phi(x)^2\right)}$$

noting that the partial derivative $\partial/\partial j$ becomes a functional derivative $\delta/\delta j(x)$.

2.4 Dodgy Commutation?

In the lectures I started from

$$\mathscr{Z}_j \triangleq \int \mathscr{D}\varphi \exp\left(i \int \mathrm{d}^4x \left(-\frac{1}{2}\varphi G^{-1}\varphi + j\varphi\right)\right)$$

and again substituted

$$\varphi'(x) \triangleq \varphi(x) + \Phi(x)$$

to get

$$\mathscr{Z}_j = \int \mathscr{D}\varphi' \exp\left(i \int \mathrm{d}^4x \left(-\frac{1}{2}\varphi' G^{-1}\varphi' - \frac{1}{2}\Phi G^{-1}\Phi + \frac{1}{2}\varphi' G^{-1}\Phi + \frac{1}{2}\Phi G^{-1}\varphi' + j\varphi' - j\Phi\right)\right).$$

I then asserted that I can write

$$\frac{1}{2}\varphi' G^{-1}\Phi \underset{dodgy?}{=} \frac{1}{2}\Phi G^{-1}\varphi'.$$

(a) The Klein Gordon action starts off life as

[1] The answers are far far simpler than "they're not renormalizable", although that's true as well.
[2] This question is done in more detail in exercise 23.7 in Lancaster and Blundell

$$S_{KG}[\varphi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \right).$$

show using integration by parts that this can be written as[3]

$$S_{KG}[\varphi] = \int \mathrm{d}^4x \left(-\frac{1}{2}\varphi\partial_\mu\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \right) = \int \mathrm{d}^4x \left(-\frac{1}{2}\varphi\left(\Box + m^2\right)\varphi \right)$$

and so $G^{-1} = \Box + m^2$.

(b) Show, again using integration by parts, that in this case

$$\int \mathrm{d}^4x \frac{1}{2}\varphi(x) G^{-1}\Phi(x) = \int \mathrm{d}^4x \frac{1}{2}\Phi(x) G^{-1}\varphi(x).$$

(c) Convince yourself by discussing with a friend that this ultimately worked because the quadratic term in the action only contained even powers of derivatives.

(d) In fact, due to our favourite Deus Ex Machina (renormalization group flow), no $(\partial\varphi)^n$ for $n > 2$ are allowed. Ignoring interaction terms φ^n $(n > 2)$ list all possible allowed combinations of φ and $\partial\varphi$. Explain why each is trivial.

(e*) An interesting exception is Chern Simons theory. This is a topological field theory - it has no kinetic term in its action, but still manages to be non-trivial. The action is

$$S[A] = \int \mathrm{d}^3x \left(\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho + j_\mu A^\mu\right)$$

with $\epsilon_{\mu\nu\rho}$ the Levi-Civita symbol (antisymmetric in all indices). Vary $A_\mu \to a_\mu + B_\mu$. This time we can do the commutation trick for a different reason. What is it? If $B_\mu(x)$ is the fixed field, what is the form of $j_\mu(x)$ which cancels the linear $a_\mu(x)$ terms?

(f) Do you think the hand-waving commutation was justified?

2.5 Differentiation

Verify using the rules of the functional derivative that

$$\left(-i\frac{\delta}{\delta j(x_2)}\right)\left(-i\frac{\delta}{\delta j(x_1)}\right)\exp\left(i\int \mathrm{d}^4x \int \mathrm{d}^4y \frac{1}{2} j(x) G(x,y) j(y)\right)\Bigg|_{j=0} = -iG(x_1, x_2).$$

3 n-point Functions

3.1 Generating Functions

In the notes it is stated that $\mathscr{Z}_j$ is a generating functional, since one can use it to generate quantities of interest by taking derivatives with respect to j (which is generally then set to zero). A simple analogy is given by Gaussian integrals. First, show that

$$I_0(\alpha) \triangleq \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\alpha x^2\right) \mathrm{d}x = \sqrt{\frac{2\pi}{\alpha}}$$

as shown in the notes. Now show that

$$I_n(\alpha) \triangleq \int_{-\infty}^{\infty} x^{2n} \exp\left(-\frac{1}{2}\alpha x^2\right) \mathrm{d}x = (2n+1)!\sqrt{2\pi}\alpha^{-1/2-n}$$

by treating I_0 as a generating function and differentiating under the integral with respect to α.

[3] Note that in a generic field theory the kinetic energy term in the Lagrangian density always takes the form $(\partial\varphi)^2$ for the relevant definitions of ∂ and φ.

3.2 More Generating Functions

There are many parallels between QFT and thermodynamics / statistical mechanics. In the latter we define the partition function to be

$$Z = \sum_i \exp(-\beta E_i)$$

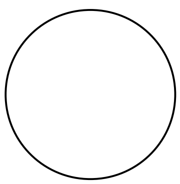
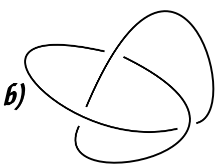
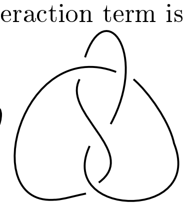
(note the similarities to the field theory definition). The partition function is a generating function for every thermodynamic function of state. By differentiating under the integral with respect to inverse temperature β show that the internal energy is given by

$$-\frac{\partial \ln Z}{\partial \beta} = \langle E \rangle \triangleq U.$$

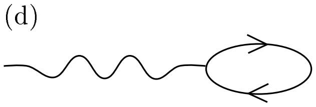
Note also that it is $\ln Z$ which is of interest here, just as $\ln \mathscr{Z}$ generated the connected diagrams in QFT (* not actually mentioned in lectures, but it's in the notes!). Find also the thermodynamic quantities C_V, C_P, and F.

3.3 Feynman-Stückelberg Interpretation

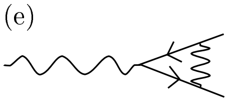
Interpret the Feynman diagrams in this picture as sequences of spacetime events. Come up with a couple of interpretations for each. What kind of interaction term is needed in each case?

a)  b)  c) 

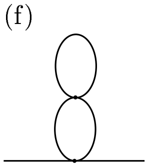
(d)



(e)



(f)

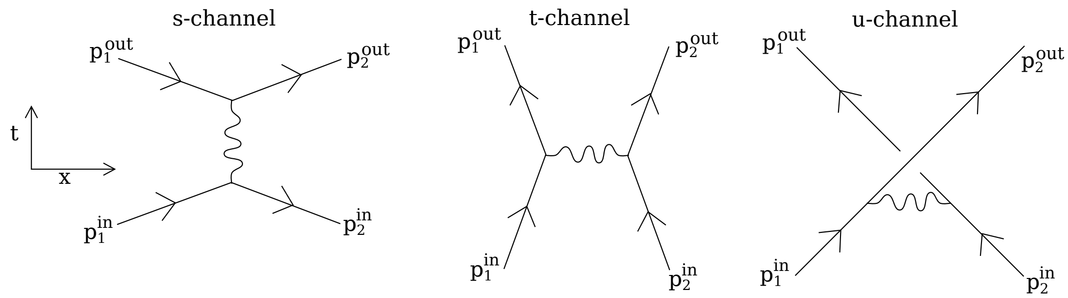


3.4 Mandelstam Variables

In 2-particle to 2-particle scattering events, where the ingoing particles have 4-momenta p_1^{in} and p_2^{in} and the outgoing particles have 4-momenta p_1^{out} and p_2^{out}, there are three well-defined relativistic quantities known as the Mandelstam variables:

$$\begin{aligned}
s &\triangleq \left(p_1^{in} + p_2^{in}\right)^2 = \left(p_1^{out} + p_2^{out}\right)^2 && \text{(CoM Energy)} \\
t &\triangleq \left(p_1^{in} - p_1^{out}\right)^2 = \left(p_2^{in} - p_2^{out}\right)^2 && \text{(4-momentum transfer)} \\
u &\triangleq \left(p_1^{in} - p_2^{out}\right)^2 = \left(p_2^{in} - p_1^{out}\right)^2 .
\end{aligned}$$

The Feynman diagrams are:



known as the s-channel, t-channel, and u-channel respectively. In this case we have picked a frame to work in such that the p^{in} particles come from the past and p^{out} go to the future. Interpret the diagrams as scattering events.

3.5 Diagrammatic Expansions

3.5.1 φ^4 partition function

Start from the partition function

$$\mathscr{W} = \mathscr{Z}_0^{-1} \int \mathscr{D}\varphi \exp\left(i \int \mathrm{d}^4x \int \mathrm{d}^4y \left(-\frac{1}{2}\varphi_y G_{xy}^{-1}\varphi_x\right) + i \int \mathrm{d}^4x \left(\frac{\lambda}{4!}\varphi_x^4\right)\right)$$

where we know that

$$\begin{aligned} \langle \varphi_y \varphi_x \rangle &= -iG_{xy} \\ \text{and } \langle \mathcal{O} \rangle &= \mathscr{Z}_0^{-1} \int \mathscr{D}\varphi \mathcal{O} \exp\left(i \int \mathrm{d}^4x \left(-\frac{1}{2}\varphi G^{-1}\varphi\right)\right). \end{aligned}$$

(a) Verify the result from the lecture, that by expanding the interaction exponential to second order in the coupling the partition function evaluates to

$$\begin{aligned} \mathscr{W} &= 1 + \frac{i\lambda}{4!} \int \mathrm{d}^4x \, (-iG_{xx})^2 \cdot 3 \\ &\quad + \frac{1}{2!}\left(\frac{i\lambda}{4!}\right)^2 \int \mathrm{d}^4x \mathrm{d}^4y \left((-iG_{xx})^2 (-iG_{yy})^2 \cdot 3^2 + (-iG_{xx})(-iG_{xy})^2(-iG_{yy}) \cdot 6^2 \cdot 2 + (-iG_{xy})^4 \cdot 4!\right). \end{aligned}$$

(b) Draw the Feynman diagrams for each term.
(c) Draw the Feynman diagrams for the terms of order λ^3.
(d) Give a couple of interpretations of each diagram in terms of virtual particle processes (virtual particles are any particles appearing as internal lines in Feynman diagrams).

3.5.2 φ^4 2-point function

(a) By again expanding the exponential as a Taylor series, evaluate the interacting 2-point function

$$\langle \varphi_2 \varphi_1 \rangle_\lambda \triangleq \left\langle \varphi_2 \varphi_1 \exp\left(i \frac{\lambda}{4!} \int \mathrm{d}^4x \varphi_x^4\right) \right\rangle$$

to order λ.
(b) Draw the corresponding Feynman diagrams.
(c) Draw all the Feynman diagrams to order λ^3.
(d) Draw all the diagrams to order λ^4, excluding those which contain vacuum bubbles.

3.5.3 φ^4 4-point function

(a) Draw the 4-point function diagrams to order λ, and then write down the corresponding analytic expressions.
(b) It turns out that only *connected* diagrams contribute to physical processes. There are two different senses of the word 'connected' used in QFT. The first means diagrams with no bubbles. The second means diagrams where all external legs are joined. Let's call them connected$_A$ and connected$_B$ (not common terminology). Selecting from the λ^2 4-point functions draw examples of $A\&B$, $A\neg B$, $B\neg A$, $\neg A\neg B$ (where $\neg$ means 'not').

3.5.4 QED-type partition function*

Let's consider a theory with two fields

$$\mathscr{W} = \mathscr{Z}_\varphi^{-1}\mathscr{Z}_\eta^{-1}\int \mathscr{D}\varphi \int \mathscr{D}\eta \exp\left(i\int \mathrm{d}^4x\int \mathrm{d}^4y\left(-\frac{1}{2}\varphi_y G_{xy}^{-1}\varphi_x - \frac{1}{2}\eta_y D_{xy}^{-1}\eta_x\right) + ig\int \mathrm{d}^4x \eta_x\varphi_x^2\right).$$

Now we have two kinds of functional average,

$$\begin{aligned}\langle \mathcal{O}\rangle_\varphi &\triangleq \mathscr{Z}_\varphi^{-1}\int \mathscr{D}\varphi \mathcal{O}\exp\left(i\int \mathrm{d}^4x\left(-\frac{1}{2}\varphi G^{-1}\varphi\right)\right)\\ \langle \mathcal{O}\rangle_\eta &\triangleq \mathscr{Z}_\eta^{-1}\int \mathscr{D}\eta \mathcal{O}\exp\left(i\int \mathrm{d}^4x\left(-\frac{1}{2}\eta D^{-1}\eta\right)\right)\end{aligned}$$

and two kinds of propagator,

$$\begin{aligned}\langle \varphi_y\varphi_x\rangle_\varphi &= -iG_{xy}\\ \langle \eta_y\eta_x\rangle_\eta &= -iD_{xy}.\end{aligned}$$

The fields are independent at the Gaussian level so the φ average does not 'see' the η field and vice versa. Denote the φ propagator by a solid line and the η propagator by a dashed line.
(a) Find the partition function $\mathscr{W}$ analytically to second order in g^2, assuming $\langle\varphi\rangle_\varphi$ and $\langle\eta\rangle_\eta$ are both zero (zero vacuum expectation value of the fields).
(b) Draw the Feynman diagrams and give some interpretation in terms of particle processes.
(b) Draw the Feynman diagrams to order g^4.

3.5.5 QED-type n-point functions*

(a) Continuing on from Section 3.5.4: the lowest order corrections to each propagator are order g^2. Draw the diagrams in each case.
(b) The simplest n-point function containing both particles as external fields is of order g:

$$\langle \varphi_3\varphi_2\eta_1\rangle_g .$$

Draw the diagram. Let's call this a 3-point function.
(b) What is the next order correction to the 3-point function? Just like in Section 3.5.3 draw examples of $A\&B$, $A\neg B$, $B\neg A$, $\neg A\neg B$.

4 Renormalization

4.1 Bad Jokes

Aside from not being funny, what is wrong with the jokes in this chapter?

4.2 Dyson Series

4.2.1 Freeman Dyson, pub Landlord

In the lectures we deduced the effect of including an infinite series of uncorrelated self-energy corrections to the Klein Gordon propagator. In this question we will reproduce the working for the electron propagator in a condensed matter setting, where the bare propagator is defined to be [4]

$$G_0(\epsilon,\mathbf{k}) = \frac{1}{\epsilon + i\delta - \xi_{\mathbf{k}}}$$

[4] This is the Euclidean form of the propagator; a 'Wick rotation' has changed variables from $i\omega_n \to \epsilon + i\delta$. For more information see Altland and Simons.

with ϵ the energy, $\xi_{\mathbf{k}}$ the energy of a state with crystal momentum $\mathbf{k}$, and δ an infinitesimal regularization $\delta \to 0^+$. First, add in self-energy corrections of the form

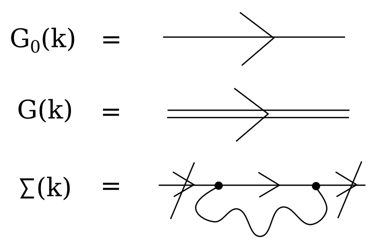
$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 \Sigma G_0 + \dots$$

and use the 'pub landlord' technique (geometric series sum) to find the renormalized propagator

$$G = \frac{1}{\epsilon + i\delta - \xi_{\mathbf{k}} - \Sigma}.$$

4.2.2 Diagrammatics

The self-energy correction comes from including the effect of some kind of interaction in the propagator. The interacting theory is re-expressed as a free theory with a different (renormalized) propagator. If we allow a QED-type interaction of electron and hole (positron in QED) annihilating to a phonon (photon in QED) we can represent the objects diagrammatically as



where it is understood that the struck-through lines indicate amputated external legs. Redo the algebraic calculation using only diagrams and the number one.

4.2.3 Spectral Function

The spectral function is defined as

$$A(\epsilon, \mathbf{k}) \triangleq -\frac{1}{\pi} \mathfrak{Im} G(\epsilon + i\delta, \mathbf{k}).$$

(a) Find the spectral function for the bare electron G_0. You should find the result is a Lorentzian which is infinitely sharp, localized on $\epsilon = \xi_{\mathbf{k}}$.
(b) The spectral function gives the probability of finding a particle with energy ϵ and crystal momentum $\mathbf{k}$. This suggests it should be positive definite, and that its integral over all energies should equal unity. Check both conditions.
(c) Now find the spectral function for the particle renormalized by a complex self-energy: $\Sigma = \Sigma' + i\Sigma''$ with Σ', Σ'' real. Assume $\Sigma'' < 0$ to avoid any confusion. By considering the effect of including Σ on the Lorentzian distribution interpret the effect of the real and imaginary parts of the self-energy on the particle's properties.
(d) Verify that the spectral function can still be interpreted as the probability for finding a particle.
(e) Discuss with a friend what is going on philosophically.

5 Effective Action

5.1 An interacting 0+0D QFT of two fields

Returning to our old friend the 0+0D QFT we will now consider two real scalar fields coupled through an interaction term:

$$\mathscr{W} = \frac{\int \mathrm{d}\phi \mathrm{d}\eta \exp\left(i\left(-\frac{a}{2}\phi^2 - \frac{b}{2}\eta^2 + g\phi^2\eta\right)\right)}{\int \mathrm{d}\phi \mathrm{d}\eta \exp\left(-\frac{i}{2}\left(a\phi^2 + b\eta^2\right)\right)}.$$

From the working of Section 2.1 we know that the bare Green's functions (2-point correlators / propagators) for the fields are

$$\begin{aligned}
\frac{\int \mathrm{d}\phi\,(\phi\phi)\exp\left(-\frac{i}{2}a\phi^2\right)}{\int \mathrm{d}\phi\exp\left(-\frac{i}{2}a\phi^2\right)} \triangleq \langle\phi\phi\rangle_\phi &= \frac{-i}{a} \\
\frac{\int \mathrm{d}\eta\,(\eta\eta)\exp\left(-\frac{i}{2}a\eta^2\right)}{\int \mathrm{d}\eta\exp\left(-\frac{i}{2}a\eta^2\right)} \triangleq \langle\eta\eta\rangle_\eta &= \frac{-i}{b}.
\end{aligned}$$

By expanding the exponential interaction term, using the working in the notes, integrate out the ϕ field to arrive at an effective action in terms of only η particles. Go to the first order beyond Gaussian in η.

5.2 Scalar QED

In the lectures we found the effective action for Scalar QED by integrating out the η field from the defining equations:

$$\begin{aligned}
\mathscr{Z} &= \mathscr{Z}_\varphi^{-1}\mathscr{Z}_\eta^{-1}\int \mathscr{D}\varphi\mathscr{D}\eta\exp\left(iS_\varphi^0 + iS_\eta^0 + iS_{int}\left[\varphi,\eta\right]\right) \\
S_\varphi^0\left[\varphi\right] &\triangleq \int \mathrm{d}^4x\int \mathrm{d}^4y\left(-\frac{1}{2}\varphi\left(x\right)A\left(x,y\right)^{-1}\varphi\left(y\right)\right) \\
S_\varphi^0\left[\eta\right] &\triangleq \int \mathrm{d}^4x\int \mathrm{d}^4y\left(-\frac{1}{2}\eta\left(x\right)B\left(x,y\right)^{-1}\eta\left(y\right)\right) \\
S_{int}\left[\varphi,\eta\right] &\triangleq \int \mathrm{d}^4x\left(g\varphi\left(x\right)\eta\left(x\right)\eta\left(x\right)\right).
\end{aligned}$$

Carry the same working out to integrate out the φ field instead. Go to the lowest non-vanishing order, and state what the next non-zero order would be.