Line Integrals

Line integrals can be a bit confusing at first, but here's a simple 5-point plan to deal with integrating any field along any crazy curve in any number of dimensions. The trick is that you can always reduce the expression to a standard integral over one parameter, which I'll call t. The plan is this:

1. Parametrise the curve by writing $\mathbf{r}(t) = (x(t), y(t), z(t))$ and state the domain of t: $t_0 \leq t < t_1$.
2. Find $\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}$, since you'll shortly require $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\mathrm{d}t$.
3. You have $\mathbf{F}(\mathbf{r})$, so write it as $\mathbf{F}(t)$ using (1).
4. You want $\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$, but rewrite it as $\int_{t_0}^{t_1} \left[\mathbf{F}(t) \cdot \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\right] \mathrm{d}t$ using the previous steps.
5. Evaluate the standard integral in (4).

Let's take as an example PS5 Q3.

> *C is the arc of the circle, centred on the origin, joining the points $(0,2)$ and $(-2,0)$. C is oriented anticlockwise. Write down a parametric description of the curve C.*
>
> *Integrate the vector field $\mathbf{F}(x,y) = (3y, -3x)$ along C.*

Step (1):

$$\begin{aligned} \mathbf{r}(t) &\triangleq \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \\ \frac{\pi}{2} &\leq t \leq \pi \end{aligned}$$

is a suitable parametrisation. It's worth remembering how to parametrise a circle.
Step (2):

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 2\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

Step (3):

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} 3y \\ -3x \end{pmatrix} = \begin{pmatrix} 6\sin(t) \\ -6\cos(t) \end{pmatrix}$$

Step (4):

$$\begin{aligned} \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\mathrm{d}t &= \int_{\frac{\pi}{2}}^{\pi} \begin{pmatrix} 6\sin(t) \\ -6\cos(t) \end{pmatrix} \cdot 2\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \mathrm{d}t \\ &= -12\int_{\frac{\pi}{2}}^{\pi} \left(\sin^2(t) + \cos^2(t)\right) \mathrm{d}t \\ &= -12\int_{\frac{\pi}{2}}^{\pi} \mathrm{d}t \end{aligned}$$

Step (5):
In this case the integral is trivially -6π.