i.e. as a geometric progression. Hence we can evaluate the following integral:

$$\begin{aligned}
\int_0^\infty \frac{x^{n-1}\,\mathrm{d}x}{z^{-1}\mathrm{e}^x - 1} &= \sum_{m=0}^{\infty} \int_0^\infty x^{n-1}((z\mathrm{e}^{-x})^{m+1}, \\
&= \sum_{m=0}^{\infty} z^{m+1} \int_0^\infty x^{n-1}\mathrm{e}^{-(m+1)x} \\
&= \sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)^n} \int_0^\infty y^{n-1}\mathrm{e}^{-y} \\
&= \Gamma(n) \sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)^n} \\
&= \Gamma(n) \sum_{k=1}^{\infty} \frac{z^k}{k^n} \\
&= \Gamma(n)\mathrm{Li}_n(z). \qquad \text{(C.34)}
\end{aligned}$$

Similarly one can show that

$$\int_0^\infty \frac{x^{n-1}\,\mathrm{d}x}{z^{-1}\mathrm{e}^x + 1} = -\Gamma(n)\mathrm{Li}_n(-z). \qquad \text{(C.35)}$$

Combining these equations, one can write in general that

$$\boxed{\int_0^\infty \frac{x^{n-1}\,\mathrm{d}x}{z^{-1}\mathrm{e}^x \pm 1} = \mp\Gamma(n)\mathrm{Li}_n(\mp z)\ .} \qquad \text{(C.36)}$$

Note that when $|z| \ll 1$, only the first term in the series in eqn C.32 contributes, and

$$\mathrm{Li}_n(z) \approx z. \qquad \text{(C.37)}$$

Note also that

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n), \qquad \text{(C.38)}$$

where $\zeta(n)$ is the Riemann zeta function (eqn C.21).

C.6 Partial derivatives

Consider x as a function of two variables y and z. This can be written $x = x(y, z)$, and we have that

$$\mathrm{d}x = \left(\frac{\partial x}{\partial y}\right)_z \mathrm{d}y + \left(\frac{\partial x}{\partial z}\right)_y \mathrm{d}z. \qquad \text{(C.39)}$$

But rearranging $x = x(y, z)$ can lead to having z as a function of x and y so that $z = z(x, y)$, in which case

$$\mathrm{d}z = \left(\frac{\partial z}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial z}{\partial y}\right)_x \mathrm{d}y. \qquad \text{(C.40)}$$

Substituting C.40 into C.39 gives

$$\mathrm{d}x = \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial x}\right)_y \mathrm{d}x + \left[\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x\right] \mathrm{d}y.$$

The terms multiplying $\mathrm{d}x$ give the **reciprocal theorem**

$$\boxed{\left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{\left(\frac{\partial z}{\partial x}\right)_y},} \tag{C.41}$$

and the terms multiplying $\mathrm{d}z$ give the **reciprocity theorem**

$$\boxed{\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.} \tag{C.42}$$

C.7 Exact differentials

An expression such as $F_1(x, y)\,\mathrm{d}x + F_2(x, y)\,\mathrm{d}y$ is known as an **exact differential** if it can be written as the differential

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}\right) \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right) \mathrm{d}y, \tag{C.43}$$

of a differentiable single-valued function $f(x, y)$. This implies that

$$F_1 = \left(\frac{\partial f}{\partial x}\right) \quad F_2 = \left(\frac{\partial f}{\partial y}\right), \tag{C.44}$$

or in vector form, $\boldsymbol{F} = \nabla f$. Hence the integral of an exact differential is path-independent, so that [where 1 and 2 are shorthands for (x_1, y_1) and (x_2, y_2)]

$$\int_1^2 F_1(x, y)\,\mathrm{d}x + F_2(x, y)\,\mathrm{d}y = \int_1^2 \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_1^2 \mathrm{d}f = f(2) - f(1), \tag{C.45}$$

and the answer depends only on the initial and final states of the system. For an **inexact differential** this is not true and knowledge of the initial and final states is not sufficient to evaluate the integral: you have to know which path was taken.

For an exact differential the integral round a closed loop is zero:

$$\oint F_1(x, y)\,\mathrm{d}x + F_2(x, y)\,\mathrm{d}y = \oint \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \oint \mathrm{d}f = 0, \tag{C.46}$$

which implies that $\nabla \times \boldsymbol{F} = 0$ (by Stokes' theorem) and hence

$$\left(\frac{\partial F_2}{\partial x}\right) = \left(\frac{\partial F_1}{\partial y}\right) \quad \text{or} \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial^2 f}{\partial y \partial x}\right). \tag{C.47}$$

For thermal physics, a crucial point to remember is that *functions of state have exact differentials.*