Advanced Quantum Physics: Problem Set 1

This problem set is due of Friday of Week 2.

Problem set marks

- Satisfactory completion of each of the four problem sets will receive 5% of the final mark for the course.
- 'Satisfactory completion' is defined as follows. You must *either* (i) make a written attempt to achieve each of the 25 marks on Question 2, *or* (ii) for any marks on Q2 you are not sure how to achieve, write a short paragraph explaining where you are stuck, listing the sections of three separate resources you have consulted.
- **Note**: the marking criteria for problem sets therefore do not require you to get any answers correct to receive full marks! In the exam you will need to get correct answers to receive the marks.
- You are expected to spend 10 hours per week on this course, including lectures and problems classes. That leaves you with around 6 hours per week to work on problem sets. The marking scheme is designed to allow you flexibility to direct your own learning: you may wish to use some of your time to revise relevant parts of other courses.

Format of problem sets

- In the exam you will need to answer four 25-mark questions in 2 hours. You will not have access to notes.
- Question 1 (worked example) and Question 2 are designed to closely mimic an exam question.
- Try to work through the worked example by yourself before looking at the answers.
- Question 3 is assorted questions to help with your understanding of the lectures. Q3 does not need to be completed to receive the marks. If you are happy with Q1 and Q2, you may prefer to complete Q3 in the problems class.

1 Lagrangian Mechanics (exam style, worked example)

Section A: mostly bookwork.

Using generalised co-ordinates $q_i(t)$ and velocities $\dot{q}_i(t)$ at time t the Lagrangian for a particle moving in a potential $V(q)$ is

$$L(q_i, \dot{q}_i) = \frac{1}{2}m\dot{q}^2 - V(q). \tag{1}$$

A.1 Write down the corresponding action governing the motion between times t_0 and t_1.

[2 marks]

A.2 Call the true trajectory of the particle $q_i(t)$. By considering variations away from this trajectory of the form $q_i(t) + \lambda\epsilon_i(t)$, formulate a mathematical statement of the principle of least action, explaining your reasoning.

[3 marks]

A.3 Using the principle of least action, derive the Euler Lagrange equations describing the motion of the particle.

[5 marks]

Section B: bringing together ideas from across the course.

Now consider a simple example of functional calculus in a different context.

The distance between two points is given by the functional

$$l[\boldsymbol{x}] = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{2}$$

where

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

B.1 Explain the origin of Eq. 2.

[3 marks]

B.2 By considering paths $\boldsymbol{x} + \lambda\boldsymbol{\epsilon}$, find a differential equation describing the path minimising the distance between x_1 and x_2. Explain all the steps in your working.

[4 marks]

B.3 Solve your equation to find the shortest distance between the two points. Does this match your intuition?

[3 marks]

Section C: more challenging.

C.1 The squared proper distance between two spacetime events in Minkowski space is

$$\mathrm{d}s^2 = \mathrm{d}x^2 - c^2\mathrm{d}t^2. \tag{4}$$

Find the path $t(x)$ that minimises the proper distance between two spacetime events.
[Hint: you may use your answers to Section B.]

[3 marks]

C.2 What is the proper distance along this path?

[2 marks]

Solutions to Question 1

Section A: mostly bookwork.

Using generalised co-ordinates $q_i(t)$ and velocities $\dot{q}_i(t)$ at time t the Lagrangian for a particle moving in a potential $V(q)$ is

$$L(q_i, \dot{q}_i, t) = \frac{1}{2} m\dot{q}^2 - V(q). \tag{5}$$

A.1 Write down the corresponding action governing the motion between times t_0 and t_1.

[2 marks]

$$S[q_i] = \int_{t_0}^{t_1} L \mathrm{d}t$$

A.2 Call the true trajectory of the particle $q_i(t)$. By considering variations away from this trajectory of the form $q_i(t) + \lambda\epsilon_i(t)$, formulate a mathematical statement of the principle of least action, explaining your reasoning.

[3 marks]

The principle of least action states that the classical trajectory of the particle extremises the action.
[1 mark]
Mathematically:

$$\left.\left(\frac{\partial S[q_i + \lambda\epsilon_i]}{\partial \lambda}\right)_{q_i, \epsilon_i}\right|_{\lambda=0} = 0$$

[1 mark].
The reason is that the principle states that the classical path extremises the action in the space of all trajectories, and an extremum by definition has a vanishing first derivative.
[1 mark]
N.B. the name is a bit of a misnomer based on the fact that 'extremises' often means 'minimises' in practice. Actually even 'extremise' is not fully correct either: the path followed is one along which the action is constant, so the variation is zero. We return to this later in the course.

A.3 Using the principle of least action, derive the Euler Lagrange equations describing the motion of the particle.

[5 marks]

$$S[q_i] = \int_{t_0}^{t_1} L(q_i, \dot{q}_i, t) \, \mathrm{d}t$$

so

$$S[q_i + \lambda\epsilon_i] = \int_{t_0}^{t_1} L(q_i + \lambda\epsilon_i, \dot{q}_i + \lambda\dot{\epsilon}_i, t) \, \mathrm{d}t$$

[1 mark].

$$\begin{aligned}
\frac{\partial S[q_i + \lambda\epsilon_i]}{\partial \lambda} &= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial q_i} \frac{\partial (q_i + \lambda\epsilon_i)}{\partial \lambda} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial (\dot{q}_i + \lambda\dot{\epsilon}_i)}{\partial \lambda} + \frac{\partial L}{\partial t} \frac{\partial t}{\partial \lambda} \right\} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial q_i} \epsilon_i + \frac{\partial L}{\partial \dot{q}_i} \dot{\epsilon}_i \right\} \mathrm{d}t
\end{aligned}$$

[1 mark].
Therefore

$$\left.\frac{\partial S\left[q_i+\lambda\epsilon_i\right]}{\partial\lambda}\right|_{\lambda=0}=0=\int_{t_0}^{t_1}\left\{\frac{\partial L}{\partial q_i}\epsilon_i+\frac{\partial L}{\partial\dot{q}_i}\dot{\epsilon}_i\right\}\mathrm{d}t.$$

[1 mark].
Integrate the second term by parts:

$$0=\int_{t_0}^{t_1}\frac{\partial L}{\partial q_i}\epsilon_i\mathrm{d}t-\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\epsilon_i\mathrm{d}t+\left[\frac{\partial L}{\partial\dot{q}_i}\epsilon_i\right]_{t_0}^{t_1}$$

but the boundary term is zero, by assumption: $\epsilon_i\left(t_0\right)=\epsilon_i\left(t_1\right)=0$.
[1 mark]
Therefore

$$0=\int_{t_0}^{t_1}\left\{\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\right\}\epsilon_i\mathrm{d}t.$$

This is true for all $\epsilon_i\left(t\right)$. The only way that can be true is if the other part of the thing inside the integral is zero. Therefore

$$\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)=0.$$

[1 mark]

Section B: bringing together ideas from across the course.

Now consider a simple example of functional calculus in a different context.

The distance between two points is given by the functional

$$l\left[\boldsymbol{x}\right]=\int_{x_0}^{x_1}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \tag{6}$$

where

$$\boldsymbol{x}=\begin{pmatrix}x\\y\end{pmatrix}. \tag{7}$$

B.2 Explain the origin of Eq. 6.

[3 marks]

The distance between two points is a functional of the path taken. Specifically, it is the integral of the line elements along the path:

$$l\left[\boldsymbol{x}\right]=\int\mathrm{d}l$$

[1 mark].
Decomposing into cartesian co-ordinates, using Pythagoras' theorem we have

$$\mathrm{d}l^2=\mathrm{d}x^2+\mathrm{d}y^2$$

or

$$\mathrm{d}l=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}$$

(+ve since lengths are +ve). Therefore

$$l\left[\boldsymbol{x}\right] = \int \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

[1 mark]
or, pulling the $\mathrm{d}x$ out,

$$l\left[\boldsymbol{x}\right] = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

[1 mark].

B.2 By considering paths $\boldsymbol{x}+\lambda\boldsymbol{\epsilon}$, find a differential equation describing the path minimising the distance between x_1 and x_2. Explain all the steps in your working.

[4 marks]

Hopefully the first part has made it clear that we seek something like the Euler Lagrange equation.

$$\begin{aligned} l\left[\boldsymbol{x}+\lambda\boldsymbol{\epsilon}\right] &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}\left(y+\lambda\epsilon_y\right)}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\ &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \end{aligned}$$

where ϵ_y is the y-component of the 2-component vector $\boldsymbol{\epsilon}$.
[1 mark]

$$\frac{\partial l\left[\boldsymbol{x}+\lambda\boldsymbol{\epsilon}\right]}{\partial\lambda} = \int_{x_0}^{x_1} \left\{ \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\right)\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\mathrm{d} \right\} x$$

[1 mark]
and

$$\left.\frac{\partial l\left[\boldsymbol{x}+\lambda\boldsymbol{\epsilon}\right]}{\partial\lambda}\right|_{\lambda=0} = \int_{x_0}^{x_1} \left\{ \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x} \right\} \mathrm{d}x = 0$$

[1 mark].
Integrating by parts,

$$0 = -\int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x}\left\{ \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \right\} \epsilon_y \mathrm{d}x + \left[\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\epsilon_y \right]_{x_0}^{x_1}$$

and the boundary term vanishes, as always, since we assume the variation vanishes at the end points. Therefore

$$0 = \int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x}\left\{ \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \right\} \epsilon_y \mathrm{d}x$$

and since this is true for all ϵ_y we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right\}=0$$

[1 mark].

B.3 Solve your equation to find the shortest distance between the two points. Does this match your intuition?

[3 marks]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right\}=0$$

so

$$\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=C$$

with C constant.
[1 mark]
Therefore

$$\begin{aligned}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= C^2\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\pm C}{\sqrt{1-C^2}}\\ y &= C_1x+C_2\end{aligned}$$

with C_i constant.
[1 mark]
This is intuitive, as the shortest distance between two points is a straight line (in Euclidean space).
[1 mark]
NB for 1 mark not all that detail is needed in the first bit; any sensible statement that the solution is linear is fine.

Section C: more challenging.

C.1 The squared proper distance between two spacetime events in Minkowski space is

$$\mathrm{d}s^2=\mathrm{d}x^2-c^2\mathrm{d}t^2. \tag{8}$$

Find the path $t(x)$ that minimises the proper distance between two spacetime events.
[Hint: you may use your answers to Section B.]

[3 marks]

The hint, combined with our correct understanding in B.2, leads us to see that

$$l\left[x_\mu\right]=\int_{x_0}^{x_1}\sqrt{1-c^2\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

[1 mark].
This is exactly the same form as before, with

$$y \to ict.$$

[1 mark]
Therefore from B.3 we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1-c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)\right\}=0$$

[1 mark]

C.2 What is the proper distance along this path?

[2 marks]

The solution to the equation is actually just as before: a straight line. Nothing has changed in the reasoning. And of course we know that the shortest proper distance between two points is obtained (only) by light:

$$x = ct$$

[1 mark].
We can therefore substitute into the 'action' to find

$$l\left[x_\mu\right]=\int_{x_0}^{x_1}\sqrt{1-c^2\left(\frac{1}{c}\right)^2}\,\mathrm{d}x=0.$$

[1 mark].
It's perhaps slightly counterintuitive, but that's the Minkowski metric for you!

2 Lagrangian Mechanics (exam style)

Section A: mostly bookwork

Consider the action

$$S[q_i] = \int_{t_i}^{t_f} L(q_i, \dot{q}_i)\,\mathrm{d}t \tag{9}$$

where the Lagrangian L depends on time only implicitly via the generalised co-ordinates $q(t)$ and velocities $\dot{q}(t)$.

A.1 Consider a general variation of the action, δS. Using Eq. 9, and the chain rule, write an expression for δS. Make it clear what is held constant in each partial derivative.

[4 marks]

A.2 Hence, or otherwise, find an expression for the functional derivative

$$\frac{\delta S[\boldsymbol{q}]}{\delta q_i(t')}.$$

Hint: You may use the fact that

$$\frac{\delta q_j(t)}{\delta q_i(t')} = \delta^i_j \delta(t - t') \tag{10}$$

where δ^i_j is the Kronecker δ, and $\delta(t-t')$ is a Dirac δ function.

[4 marks]

A.3 Hence explain the origin of the Euler Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{11}$$

[2 marks]

Section B: bringing together ideas from across the course

In relativistic problems space and time must be treated on equal footing. The action can be written in terms of a Lagrangian density $\mathcal{L}$:

$$S[\varphi] = \int \mathrm{d}t L = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\mathcal{L}(\varphi, \partial_\mu \varphi). \tag{12}$$

Here $\varphi(x^\mu)$ is a function of the spacetime co-ordinates

$$x^\mu = \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix}^\mu \tag{13}$$

and

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial \boldsymbol{x}} \end{pmatrix}^\mu. \tag{14}$$

B.1 By considering the two variables of which the Lagrangian density is an explicit function, write an expression for the variation of the action δS.

[4 marks]

B.2 By setting

$$\frac{\delta S\left[\varphi\right]}{\delta\varphi\left(x^{\nu}\right)}=0 \tag{15}$$

derive the relativistic Euler Lagrange equations.
Hint: you may use the fact that

$$\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}=\delta\left(x^{\mu}-x^{\nu}\right). \tag{16}$$

[4 marks]

B.3 Consider the action

$$S\left[\varphi\right]=\int \mathrm{d}t\int \mathrm{d}^{3}\boldsymbol{x}\left\{\frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi-\frac{m^{2}c^{2}}{2\hbar^{2}}\varphi^{2}\right\} \tag{17}$$

where a sum over repeated indices μ is assumed. Show that the Euler Lagrange equation for this action is the Klein Gordon equation

$$\left(\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}-\nabla^{2}+\frac{m^{2}c^{2}}{\hbar^{2}}\right)\varphi=0. \tag{18}$$

Hint: you may use the fact that

$$\partial_{\mu}\partial^{\mu}=\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}-\nabla^{2}. \tag{19}$$

[2 marks]

Section C: more challenging

C.1 By writing the function φ as

$$\varphi\left(ct,\boldsymbol{x}\right)=\phi\left(ct,\boldsymbol{x}\right)\exp\left(-imc^{2}t/\hbar\right) \tag{20}$$

where mc^{2} is the rest mass of the particle, show that the Klein Gordon equation reduces to the Schroedinger equation in the non-relativistic limit. State any assumptions that you make.

[5 marks]

3 Lagrangian Mechanics (assorted questions not in exam style)

NB you do not need to submit answers to these questions. Marks are provided simply to offer guidance on how long to spend on each question.
If you are happy with your answers to Q2, you might choose to work on Q3 in the problems class.

3.1 Connecting classical to quantum mechanics

The Hamiltonian in classical mechanics can be found from the Lagrangian using a Legendre transform:

$$H = p_i \dot{q}^i - L \tag{21}$$

where

$$p_i \triangleq \frac{\partial L}{\partial \dot{q}^i}. \tag{22}$$

Hamilton's equations of motion are then

$$\dot{q} = \frac{\partial H}{\partial p}; \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{23}$$

Finally, the Poisson bracket is defined as

$$\{f, g\} \triangleq \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i}. \tag{24}$$

3.11 What is held constant in each of the seven partial derivatives?

[7 marks]

3.12 Prove the relations given in the lectures:

$$\dot{q}_i = \{q_i, H\} \tag{25}$$

$$\dot{p}_i = \{p_i, H\}. \tag{26}$$

[4 marks]

3.13 Assume we have a function with the dependence $f(q_i, p_i, t)$. Derive Hamilton's equation of motion:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \left(\frac{\partial f}{\partial t}\right)_{q_i, p_i}. \tag{27}$$

[4 marks]

3.14 Explain the difference between the Heisenberg and Schroedinger pictures of quantum mechanics. You may need to remind yourself of the third year quantum course.

[4 marks]

3.15 Using the labels H and S to label the two pictures, we have the following relationship between operators:

$$\hat{A}_H(t) = e^{-i\hat{H}t/\hbar}\hat{A}_S(t)\, e^{i\hat{H}t/\hbar} \tag{28}$$

where the operator in the Schroedinger picture has an explicit time dependence (which is rarely a case we consider). Assuming the Hamiltonian is time independent, derive the Heisenberg equation of motion:

$$\frac{\mathrm{d}\hat{A}_H}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{A}_H, \hat{H}\right] + \left(\frac{\partial \hat{A}_S}{\partial t}\right)_H$$

explaining the meaning of the final term.

[4 marks]

3.17 Using the Heisenberg equation of motion, prove Ehrenfest's theorem:

$$\frac{\mathrm{d}\left\langle \hat{A}\right\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\left\langle\left[\hat{A}, \hat{H}\right]\right\rangle + \left\langle\frac{\partial \hat{A}}{\partial t}\right\rangle \tag{29}$$

where

$$\left\langle \hat{A}\right\rangle \triangleq \langle\psi|\hat{A}|\psi\rangle \tag{30}$$

for an arbitrary state $|\psi\rangle$.

[2 marks]

3.18 Why were we able to drop the H subscript in 3.17?

[1 mark]

3.19 Find a friend and discuss the relationship between classical constants of motion, and good quantum numbers.