AQM 2023 PS1 Answers

October 12, 2023

2 Lagrangian Mechanics (exam style)

Section A: mostly bookwork

Consider the action

$$S[q_i] = \int_{t_i}^{t_f} L(q_i, \dot{q}_i)\,\mathrm{d}t \tag{1}$$

where the Lagrangian L depends on time only implicitly via the generalised co-ordinates $q(t)$ and velocities $\dot{q}(t)$.

A.1 Consider a general variation of the action, δS. Using Eq. 1, and the chain rule, write an expression for δS. Make it clear what is held constant in each partial derivative.

[4 marks]

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f} \delta L(q_i, \dot{q}_i, t)\,\mathrm{d}t \\ &= \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \delta \dot{q}_i \right\} \mathrm{d}t\end{aligned}$$

[1 mark] each for the two terms
[1 mark] each for the two pairs held constant. Note: holding t constant is unnecessary as there is no explicit time dependence, so the mark is received whether it is there or not.

A.2 Hence, or otherwise, find an expression for the functional derivative

$$\frac{\delta S[\boldsymbol{q}]}{\delta q_i(t')}.$$

Hint: You may use the fact that

$$\frac{\delta q_j\left(t\right)}{\delta q_i\left(t'\right)} = \delta^i_j \delta\left(t - t'\right) \tag{2}$$

where δ^i_j is the Kronecker δ, and $\delta\left(t - t'\right)$ is a Dirac δ function.

[4 marks]

$$\delta S = \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \delta \dot{q}_i \right\} \mathrm{d}t$$

$$\frac{\delta S}{\delta q_i\left(t'\right)} = \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \frac{\delta q_i\left(t\right)}{\delta q_i\left(t'\right)} + \left(\frac{\partial L}{\partial \dot{q}}\right)_{q,t} \frac{\delta \dot{q}\left(t\right)}{\delta q_i\left(t'\right)} \right\} \mathrm{d}t$$

[1 mark]
2nd term by parts:

$$\frac{\delta S}{\delta q_j\left(t'\right)} = \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \frac{\delta q_i\left(t\right)}{\delta q_j\left(t'\right)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \frac{\delta q_i\left(t\right)}{\delta q_j\left(t'\right)} \right\} \mathrm{d}t + \left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \frac{\delta q_i\left(t\right)}{\delta q_j\left(t'\right)} \right]_{t_i}^{t_f}$$

[1 mark]
and the boundary term is assumed zero because of the boundary conditions.
Then we use the stated expression:

$$\frac{\delta S}{\delta q_j\left(t'\right)} = \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \right\} \delta^j_i \delta(t - t') \mathrm{d}t$$

[1 mark]
Giving the result

$$\frac{\delta S}{\delta q_j\left(t'\right)} = \left(\frac{\partial L}{\partial q_j}\right)_{\dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\partial L}{\partial \dot{q}_j}\right)_{q}$$

[1 mark]

A.3 Hence explain the origin of the Euler Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{3}$$

[2 marks]

The principle of least action says that classical paths extremise the action.
[1 mark]
Hence the l.h.s. is 0, and the equations follow.
[1 mark]

Section B: bringing together ideas from across the course

In relativistic problems space and time must be treated on equal footing. The action can be written in terms of a Lagrangian density $\mathcal{L}$:

$$S[\varphi] = \int \mathrm{d}t L = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \mathcal{L}(\varphi, \partial_\mu \varphi). \tag{4}$$

Here $\varphi(x^\mu)$ is a function of the spacetime co-ordinates

$$x^\mu = \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix}^\mu \tag{5}$$

and

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial \boldsymbol{x}} \end{pmatrix}^\mu. \tag{6}$$

B.1 By considering the two variables of which the Lagrangian density is an explicit function, write an expression for the variation of the action δS. Make it clear what is held constant in each partial derivative.

[4 marks]

$$\begin{aligned} \delta S &= \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \delta\mathcal{L}(\varphi, \partial_\mu\varphi) \\ &= \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} \delta\varphi + \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)_{\varphi} \delta\partial_\mu\varphi \right\}. \end{aligned}$$

As before, **[1 mark]** for each term and **[1 mark]** for each correct statement as to what is held constant.

B.2 By setting

$$\frac{\delta S[\varphi]}{\delta\varphi(x^\nu)} = 0 \tag{7}$$

derive the relativistic Euler Lagrange eqautions.

Hint: you may use the fact that

$$\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}=\delta\left(x^{\mu}-x^{\nu}\right). \tag{8}$$

[4 marks]

$$\frac{\delta S}{\delta\varphi\left(x^{\nu}\right)}=\int\mathrm{d}t\int\mathrm{d}^{3}\boldsymbol{x}\left\{\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_{\partial_{\mu}\varphi}\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}+\left(\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\right)_{\varphi}\frac{\delta\partial_{\mu}\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}\right\}$$

[1 mark]

$$0=\int\mathrm{d}t\int\mathrm{d}^{3}\boldsymbol{x}\left\{\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_{\partial_{\mu}\varphi}\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}+\left(\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\right)_{\varphi}\frac{\delta\partial_{\mu}\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}\right\}$$

[1 mark]. Integrate the second term by parts:

$$0=\int\mathrm{d}t\int\mathrm{d}^{3}\boldsymbol{x}\left\{\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_{\partial_{\mu}\varphi}\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\right)_{\varphi}\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}\right\}+\left[\left(\boldsymbol{\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}}\right)_{\boldsymbol{\varphi}}\boldsymbol{\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}}\right]_{\boldsymbol{x_i^{\mu}}}^{\boldsymbol{x_f^{\mu}}} \tag{9}$$

and the boundary term disappears by assumption.
[1 mark]
Use the stated relation to find

$$0=\int\mathrm{d}t\int\mathrm{d}^{3}\boldsymbol{x}\left\{\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_{\partial_{\mu}\varphi}-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\right)_{\varphi}\right\}\delta\left(x^{\mu}-x^{\nu}\right)$$
$$0=\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_{\partial_{\mu}\varphi}-\partial_{\nu}\left(\frac{\partial\mathcal{L}}{\partial\partial_{\nu}\varphi}\right)_{\varphi}$$

[1 mark].

B.3 Consider the action

$$S\left[\varphi\right]=\int\mathrm{d}t\int\mathrm{d}^{3}\boldsymbol{x}\left\{\frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi-\frac{m^{2}c^{2}}{2\hbar^{2}}\varphi^{2}\right\} \tag{10}$$

where a sum over repeated indiced μ is assumed. Show that the Euler Lagrange equation for this action is the Klein Gordon equation

$$\left(\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}-\nabla^{2}+\frac{m^{2}c^{2}}{\hbar^{2}}\right)\varphi=0. \tag{11}$$

Hint: you may use the fact that

$$\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{12}$$

[2 marks]

$$0 = \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi}\right)_{\varphi}$$
$$0 = -\frac{m^2c^2}{\hbar^2}\varphi - \partial_\nu \partial^\nu \varphi$$

[1 mark] and so

$$\left(\partial_\nu \partial^\nu + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0$$
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0$$

[1 mark].

Section C: more challenging

C.1 By writing the function φ as

$$\varphi\left(ct, \boldsymbol{x}\right) = \phi\left(ct, \boldsymbol{x}\right)\exp\left(-imc^2t/\hbar\right) \tag{13}$$

where mc^2 is the rest mass of the particle, show that the Klein Gordon equation reduces to the Schroedinger equation in the non-relativistic limit. State any assumptions that you make.

[5 marks]

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0 \tag{14}$$
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi\exp\left(-imc^2t/\hbar\right) = 0 \tag{15}$$

[1 mark]. Focus on the time derivative:

$$\frac{\partial^2}{\partial t^2}\left(\phi \exp\left(-imc^2 t/\hbar\right)\right) = \frac{\partial}{\partial t}\left(\dot{\phi}\exp\left(-imc^2 t/\hbar\right) - \frac{imc^2}{\hbar}\phi\exp\left(-imc^2 t/\hbar\right)\right) \tag{16}$$

$$= \left\{\ddot{\phi} - 2\frac{imc^2}{\hbar}\dot{\phi} + \left(\frac{imc^2}{\hbar}\right)^2\phi\right\}\exp\left(-imc^2 t/\hbar\right). \tag{17}$$

[1 mark]
In the non-relativistic limit, mc^2 is the dominant term, so drop the $\ddot{\phi}$ term.
[1 mark]
This gives

$$\left(-2\frac{im}{\hbar}\frac{\partial}{\partial t} - \frac{m^2c^2}{\hbar^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi\exp\left(-imc^2 t/\hbar\right) = 0$$

[1 mark]. Rewriting,

$$i\hbar\frac{\partial}{\partial t}\varphi = -\frac{\hbar^2}{2m}\nabla^2\varphi$$

as required
[1 mark].