Advanced Quantum Physics: Problem Set 2

This problem set is due of Friday of Week 3.

Problem set marks

- Satisfactory completion of each of the four problem sets will receive 5% of the final mark for the course.
- 'Satisfactory completion' is defined as follows. You must *either* (i) make a written attempt to achieve each of the 25 marks on Question 2, *or* (ii) for any marks on Q2 you are not sure how to achieve, write a short paragraph explaining where you are stuck, listing the sections of three separate resources you have consulted.
- **Note**: the marking criteria for problem sets therefore do not require you to get any answers correct to receive full marks! In the exam you will need to get correct answers to receive the marks.
- You are expected to spend 10 hours per week on this course, including lectures and problems classes. That leaves you with around 6 hours per week to work on problem sets. The marking scheme is designed to allow you flexibility to direct your own learning: you may wish to use some of your time to revise relevant parts of other courses.

Format of problem sets

- In the exam you will need to answer four 25-mark questions in 2 hours. You will not have access to notes.
- Question 1 (worked example) and Question 2 are designed to closely mimic an exam question.
- Try to work through the worked example by yourself before looking at the answers.

1 Path Integral Quantum Mechanics (exam style, worked example)

Section A: mostly bookwork.

A.1 The time evolution operator $\hat{U}(t-t_0)$ is defined by

$$|\psi(t)\rangle = \hat{U}(t-t_0)\,|\psi(t_0)\rangle. \tag{1}$$

Show that for probability to be conserved, $\hat{U}$ must be unitary.

[4 marks]

A.2 A particle is initially at position x_0 at time t_0. Explain why the amplitude to find the particle at position x at time T is given by the propagator

$$K(x, T; x_0, t_0). \tag{2}$$

[6 marks]

Section B: bringing together ideas from across the course.

B.1 A particle starts at position $x = 0$ at time $t = 0$. At time t' it travels through a slit of width $2b$ centred about $x = 0$ before reaching a screen at time T. Explain with the aid of a diagram why the amplitude to find the particle at position x on the screen is given by

$$\psi(x, T) = \int_{-b}^{b} \mathrm{d}x' K(x, T; x', t')\, K(x', t'; 0, 0). \tag{3}$$

[4 marks]

B.2 The propagator in free space is given by

$$K(x_f, t_f; x_i, t_i) = \sqrt{\frac{m}{2\pi\hbar i}} \frac{1}{\sqrt{t_f - t_i}} \exp\left(i\frac{m(x_f - x_i)^2}{2\hbar(t_f - t_i)}\right).$$

Show that the amplitude to find the particle in **B.1** at position x at time T is given by

$$\psi(x, T) = \frac{1}{2\pi i} \exp\left(i\frac{mx^2}{2\hbar T}\right) \sqrt{\frac{2m}{\hbar T}} \int_{-a(x)}^{a(x)} \mathrm{d}y \exp\left(iy^2\right)$$

for a suitably defined $a(x)$.

[6 marks]

Section C: more challenging.

The *Fresnel integrals* are defined as

$$C(x) \triangleq \int_0^x \mathrm{d}y \cos\left(y^2\right)$$
$$S(x) \triangleq \int_0^x \mathrm{d}y \sin\left(y^2\right).$$

C.1 Show that the intensity at position x on the screen can be written as

$$I(x, T) = \frac{2m}{\hbar\pi^2 T}\left(C^2(a) + S^2(a)\right).$$

Hint: You will need to use the fact that both $C(x)$ and $S(x)$ are odd functions.

[5 marks]

Path Integral Quantum Mechanics: Solutions

Section A: mostly bookwork.

A.1 The time evolution operator $\hat{U}(t - t_0)$ is defined by

$$|\psi(t)\rangle = \hat{U}(t - t_0)|\psi(t_0)\rangle. \tag{4}$$

Show that for probability to be conserved, $\hat{U}$ must be unitary.

[4 marks]

Probability is the square modulus of the amplitude (the Born rule).
[1 mark]
Therefore we require

$$|\langle\psi(t)|\psi(t)\rangle|^2 = |\langle\psi(t_0)|\psi(t_0)\rangle|^2$$

[1 mark]
and

$$\langle\psi(t)|\psi(t)\rangle = \exp(i\phi)\langle\psi(t_0)|\psi(t_0)\rangle$$

where ϕ is some real phase. However, we know what this phase is, since a proper normalisation of a state is defined as

$$\langle\psi|\psi\rangle = 1$$

and so $\phi = 0$. If the phase isn't commented on, that's fine!
From the stated equation,

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(t_0)|\hat{U}^\dagger\hat{U}|\psi(t_0)\rangle$$

[1 mark].
Therefore we require

$$\hat{U}^\dagger\hat{U} = \hat{\mathbb{I}}$$

which is the definition of unitarity.
[1 mark]

A.2 A particle is initially at position x_0 at time t_0. Explain why the amplitude to find the particle at position x at time T is given by the propagator

$$K(x, T; x_0, t_0). \tag{5}$$

[6 marks]

This is a long derivation, but it's from the notes. First project the equation given in A1 to the position basis:

$$\langle x|\psi(T)\rangle = \langle x|\hat{U}(T - t_0)|\psi(t_0)\rangle$$

[1 mark].
Now insert a decomposition of the identity into the position basis:

$$\hat{\mathbb{I}} = \int \mathrm{d}x' |x'\rangle\langle x'|$$

[1 mark]
to give

$$\langle x|\psi(T)\rangle = \int \mathrm{d}x' \langle x|\hat{U}(T - t_0)|x'\rangle\langle x'|\psi(t_0)\rangle$$

[1 mark].
The propagator is defined to be

$$K(x, T; x', t_0) = \langle x|\hat{U}(T - t_0)|x'\rangle$$

and so we have

$$\langle x|\psi(T)\rangle = \int \mathrm{d}x' K(x, T; x', t_0) \langle x'|\psi(t_0)\rangle$$

[1 mark]
Finally, since the particle was stated to be initially at a definite position x_0, it must have been described at the Dirac delta function at that instant:

$$\langle x'|\psi(t_0)\rangle = \delta(x' - x_0)$$

[1 mark]
giving

$$\langle x|\psi(T)\rangle = \int \mathrm{d}x' K(x, T; x', t_0)\, \delta(x' - x_0)$$
$$\psi(x, T) = K(x, T; x_0, t_0)$$

as required.
[1 mark]

Section B: bringing together ideas from across the course.

B.1 In Fig. 1 A particle starts at position $x = 0$ at time $t = 0$. At time t' it travels through a slit of width $2b$ centred about $x = 0$ before reaching a screen at time T. Explain with the aid of a diagram why the amplitude to find the particle at position x on the screen is given by

$$\psi(x, T) = \int_{-b}^{b} \mathrm{d}x' K(x, T; x', t') K(x', t'; 0, 0). \tag{6}$$

[4 marks]

In quantum mechanics amplitudes play the role of probabilities in classical problems. In particular,

$$\mathrm{Amp}(A \text{ and } B) = \mathrm{Amp}(A)\, \mathrm{Amp}(B)$$
$$\mathrm{Amp}(A \text{ or } B) = \mathrm{Amp}(A) + \mathrm{Amp}(B).$$

For the particle to reach point x on the screen, it must first reach point x' within the slit: this is a case of 'A and B', with A =(particle reaches x, T) and B =(particle reaches x', t'). But this is true for all allowed positions x' within the slit, so we must sum over these possibilities. This is a case of 'A or B' (or C or D...), with A, B, etc labelling all the points within the slit.
[2 marks] for any reasonable explanation

[2 marks] for a decent picture with relevant points labelled.

B.2 The propagator in free space is given by

$$K(x_f, t_f; x_i, t_i) = \sqrt{\frac{m}{2\pi\hbar i}} \frac{1}{\sqrt{t_f - t_i}} \exp\left(i \frac{m(x_f - x_i)^2}{2\hbar(t_f - t_i)}\right).$$

Show that the amplitude to find the particle in **B.1**. at position x at time T is given by

$$\psi(x, T) = \frac{1}{2\pi i} \exp\left(i \frac{mx^2}{2\hbar T}\right) \sqrt{\frac{2m}{\hbar T}} \int_{-a(x)}^{a(x)} \mathrm{d}y \exp\left(iy^2\right)$$

for a suitably defined $a(x)$.

[6 marks]

I'll be honest here: the algebra required for this question is much harder than anything you would realistically be required to perform in an exam without a lot more help. I wanted to give you exam-style questions so you have some idea what to expect, but I also wanted to cover relevant material thoroughly. So let's give it a go, but don't worry if it looks like too much – it is!

$$\begin{aligned}\psi(x, T) &= \int_{-b}^{b} \mathrm{d}x' K(x, T; x', t') K(x', t'; 0, 0) \\ &= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'(T - t')}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar}\left(\frac{(x - x')^2}{(T - t')} + \frac{x'^2}{t'}\right)\right)\end{aligned}$$

[1 mark]

$$\begin{aligned}\psi(x, T) &= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'(T - t')}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar(T - t')}\left((x - x')^2 + x'^2 \frac{T - t'}{t'}\right)\right) \\ &= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'(T - t')}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar(T - t')}\left(x^2 + x'^2 - 2xx' + x'^2\left(\frac{T}{t'} - 1\right)\right)\right) \\ &= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'(T - t')}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar(T - t')}\left(\frac{T}{t'}\right)\left(x'^2 - \left(\frac{t'}{T}\right)2xx' + \left(\frac{t'}{T}\right)x^2\right)\right)\end{aligned}$$

Now complete the square:

$$\psi(x, T) = \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'(T - t')}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar(T - t')}\left(\frac{T}{t'}\right)\left(\left(x' - \left(\frac{t'}{T}\right)x\right)^2 + \frac{t'}{T}\left(1 - \frac{t'}{T}\right)x^2\right)\right)$$

[2 marks]
The x^2 term is not a function of x' so pulls out of the intergral:

$$\psi(x, T) = \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'(T - t')}} \exp\left(i \frac{mx^2}{2\hbar T}\right) \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar(T - t')} \frac{T}{t'}\left(x' - \left(\frac{t'}{T}\right)x\right)^2\right)$$

[1 mark]
Now change variables:

$$y' = \sqrt{\frac{m}{2\hbar(T - t')} \frac{T}{t'}} x'$$

and define

$$b' = \sqrt{\frac{m}{2\hbar\left(T - t'\right)}\frac{T}{t'}}b$$

giving

$$\psi\left(x, T\right) = \frac{1}{2\pi i}\exp\left(i\frac{mx^2}{2\hbar T}\right)\sqrt{\frac{2m}{\hbar T}}\int_{-b'}^{b'}\mathrm{d}y'\exp\left(i\left(y' - \sqrt{\frac{m}{2\hbar\left(T - t'\right)}\frac{T}{t'}}\left(\frac{t'}{T}\right)x\right)^2\right)$$

And change variables again to remove the x term in the integrand:

$$y = y' - \sqrt{\frac{m}{2\hbar\left(T - t'\right)}\frac{T}{t'}}\left(\frac{t'}{T}\right)x$$

[1 mark]
defining

$$\begin{aligned} a\left(x\right) &= b' - \sqrt{\frac{m}{2\hbar\left(T - t'\right)}\frac{T}{t'}}\left(\frac{t'}{T}\right)x \\ &= \sqrt{\frac{m}{2\hbar\left(T - t'\right)}\frac{T}{t'}}\left(b - \left(\frac{t'}{T}\right)x\right) \end{aligned}$$

[1 mark]
gives

$$\psi\left(x, T\right) = \frac{1}{2\pi i}\exp\left(i\frac{mx^2}{2\hbar T}\right)\sqrt{\frac{2m}{\hbar T}}\int_{-a}^{a}\mathrm{d}y\exp\left(iy^2\right).$$

[I wouldn't be surprised if I've made some errors in my own algebra here; please let me know if you disagree. As a basic check, ψ does at least have the correct units.]

Section C: more challenging.

The *Fresnel integrals* are defined as

$$\begin{aligned} C\left(x\right) &\triangleq \int_0^x \mathrm{d}y\cos\left(y^2\right) \\ S\left(x\right) &\triangleq \int_0^x \mathrm{d}y\sin\left(y^2\right). \end{aligned}$$

C.1 Show that the intensity at position x on the screen can be written as

$$I\left(x, T\right) = \frac{2m}{\hbar\pi^2 T}\left(C^2\left(a\right) + S^2\left(a\right)\right).$$

Hint: You will need to use the fact that both $C\left(x\right)$ and $S\left(x\right)$ are odd functions.

[5 marks]

$$
\begin{aligned}
\int_{-a}^{a} \mathrm{d}y \exp\left(iy^2\right) &= \int_{0}^{a} \mathrm{d}y \exp\left(iy^2\right) + \int_{-a}^{0} \mathrm{d}y \exp\left(iy^2\right) \\
&= \int_{0}^{a} \mathrm{d}y \exp\left(iy^2\right) - \int_{0}^{-a} \mathrm{d}y \exp\left(iy^2\right) \\
&= C\left(a\right) + iS\left(a\right) - C\left(-a\right) - iS\left(-a\right) \\
&= 2\left(C\left(a\right) + iS\left(a\right)\right)
\end{aligned}
$$

where the last line used that the functions are both odd.
[2 marks]
Therefore

$$
\psi\left(x,T\right) = \frac{1}{\pi i} \exp\left(i\frac{mx^2}{2\hbar T}\right)\sqrt{\frac{2m}{\hbar T}}\left(C\left(a\right) + iS\left(a\right)\right).
$$

[1 mark]

$$
I\left(x,T\right) = \left|\psi\left(x,T\right)\right|^2
$$

[1 mark]
and so

$$
I\left(x,T\right) = \frac{2m}{\hbar\pi^2 T}\left(C^2\left(a\right) + S^2\left(a\right)\right).
$$

[1 mark]

2 Path Integral Quantum Mechanics (exam style)

Section A: mostly bookwork.

In this question you will calculate the amplitude for a quantum particle at an initial positon x_i at time t_i to move to position x_f at time t_f, in the special case $V = 0$.

A.1 Explain the meaning of the following terms, giving mathematical expressions in each case.

(i) The time evolution operator $\hat{U}\left(t_f - t_i\right)$

[3 marks]

(ii) The propagator $K\left(x_f, t_f; x_i, t_i\right)$.

[3 marks]

A.2 For a particle in free space, $V = 0$, explain why

$$K\left(x_f, t_f; x_i, t_i\right) = \langle x_f | \exp\left(-i\hat{p}^2\left(t_f - t_i\right)/2m\hbar\right) | x_i \rangle \tag{7}$$

where $\hat{p}$ is the momentum operator and m is the mass of the particle.

[4 marks]

Section B: bringing together ideas from across the course.

B.1 Define the Gaussian integral

$$I\left(a\right) = \int_{-\infty}^{\infty} \exp\left(-ax^2\right) \mathrm{d}x. \tag{8}$$

By considering I^2, and working in plane polar co-ordinates, show that

$$I\left(a\right) = \sqrt{\frac{\pi}{a}}. \tag{9}$$

[5 marks]

B.2 Now consider the Gaussian integral

$$I\left(a, b\right) = \int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) \mathrm{d}x. \tag{10}$$

By completing the square, or otherwise, show that

$$I\left(a, b\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right). \tag{11}$$

[3 marks]

B.3 By resolving the identity operator in the momentum basis, or otherwise, show that

$$\exp\left(-i\hat{p}^2\left(t_f - t_i\right)/2m\right) = \int_{-\infty}^{\infty} \mathrm{d}p \exp\left(-ip^2\left(t_f - t_i\right)/2m\right) |p\rangle\langle p| \tag{12}$$

where

$$\hat{p}|p\rangle = p|p\rangle. \tag{13}$$

[2 marks]

Section C: more challenging.

C.1 Show that the propagator in free space is given by

$$K\left(x_f, t_f; x_i, t_i\right) = \sqrt{\frac{m}{2\pi i\hbar}} \frac{1}{\sqrt{t_f - t_i}} \exp\left(\frac{i\left(x_f - x_i\right)^2 m}{2\hbar\left(t_f - t_i\right)}\right). \tag{14}$$

Hint: you will need to use the answers to A2, B2, and B3. You may use the fact that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(ipx\right). \tag{15}$$

[5 marks]