Advanced Quantum Physics: Problem Set 3

This problem set is due of Friday of Week 4.

Problem set marks

- Satisfactory completion of each of the four problem sets will receive 5% of the final mark for the course.
- 'Satisfactory completion' is defined as follows. You must *either* (i) make a written attempt to achieve each of the 25 marks on Question 2, *or* (ii) for any marks on Q2 you are not sure how to achieve, write a short paragraph explaining where you are stuck, listing the sections of three separate resources you have consulted.
- **Note**: the marking criteria for problem sets therefore do not require you to get any answers correct to receive full marks! In the exam you will need to get correct answers to receive the marks.
- You are expected to spend 10 hours per week on this course, including lectures and problems classes. That leaves you with around 6 hours per week to work on problem sets. The marking scheme is designed to allow you flexibility to direct your own learning: you may wish to use some of your time to revise relevant parts of other courses.

Format of problem sets

- In the exam you will need to answer four 25-mark questions in 2 hours. You will not have access to notes.
- Question 1 (worked example) and Question 2 are designed to closely mimic an exam question.
- Try to work through the worked example by yourself before looking at the answers.
- Question 3 is assorted questions to help with your understanding of the lectures. Q3 does not need to be completed to receive the marks. If you are happy with Q1 and Q2, you may prefer to complete Q3 in the problems class.

1 Semiclassics (worked example)

Section A: mostly bookwork.

A.1 Explain why the action for a particle in a simple harmonic oscillator, between times $t = 0$ and $t = T$, is given by

$$S[x] = \frac{1}{2}m \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right). \tag{1}$$

[2 marks]

A.2 Show that classical trajectories obey

$$\ddot{x} = -\omega^2 x.$$

[5 marks]

A.3 Solve for the classical trajectory $x(t)$ assuming $x(t = 0) = 0$ and $x(t = T) = X$.

[3 marks]

Section B: bringing together ideas from across the course.

B.1 Explain the method of stationary phase.

[2 marks]

B.2 Find an expression for the normalisation $\mathcal{Z}^{-1}$ in terms of a Gaussian functional integral (which you do not need to evaluate).

[3 marks]

B.3 Using the method of stationary phase and your answer to **A.3**, find an approximate expression for the propagator of the harmonic oscillator. You do not need to evaluate the normalisation $\mathcal{Z}$.

[5 marks]

Section C: more challenging.

Now consider the harmonic oscillator forced with a time-independent force F, for which the action is

$$S[x] = \int_0^T \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - Fx\right) \mathrm{d}t'. \tag{2}$$

C.1 Show that the propagator is

$$K_{forced}(X, T; 0, 0) = \exp(i\epsilon T/\hbar)\, K(X, T; 0, 0)$$

where ϵ is an energy you should find.

[5 marks]

Semiclassics (worked example solutions)

Section A: mostly bookwork.

A.1 Explain why the action for a particle in a simple harmonic oscillator, between times $t = 0$ and $t = T$, is given by

$$S[x] = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right). \tag{3}$$

[2 marks]

The action in general is

$$S = \int_0^T \mathrm{d}t' L(x, \dot{x}) = \int_0^T \mathrm{d}t' \left(\frac{1}{2} m \dot{x}^2 - V(x)\right) \tag{4}$$

(excluding magnetic fields, curved spaces, and so on, that we haven't met yet and which are irrelevant here).
[1 mark]
The harmonic oscillator potential is

$$V(x) = \frac{1}{2} m \omega^2 x^2$$

[1 mark]
giving the desired result.

A.2 Show that classical trajectories obey

$$\ddot{x} = -\omega^2 x.$$

[5 marks]

Classical trajectories are those for which the action is stationary.
[1 mark]

$$S[x] = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right)$$

I will use functional derivatives, but any method is fine. Other choices are to explicitly vary the trajectory $x \to x + \lambda \epsilon$, or to quote the general Euler Lagrange equation and insert the Lagrangian.

$$\frac{\delta S}{\delta x(t)} = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(2\dot{x} \frac{\delta \dot{x}(t')}{\delta x(t)} - 2\omega^2 x \frac{\delta x(t')}{\delta x(t)}\right)$$

[1 mark]
and integrate the $\dot{x}$ term by parts:

$$\frac{\delta S}{\delta x(t)} = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(-2\ddot{x} \frac{\delta x(t')}{\delta x(t)} - 2\omega^2 x \frac{\delta x(t')}{\delta x(t)}\right)$$

(the boundary term vanishes by construction, as usual).
[1 mark]
Now use

$$\frac{\delta x(t')}{\delta x(t)} = \delta(t - t')$$

to give

$$\begin{aligned}\frac{\delta S}{\delta x(t)} &= -m\int_0^T \mathrm{d}t'\,\delta(t-t')\left(\ddot{x}+\omega^2 x\right)\\ &= -m\left(\ddot{x}+\omega^2 x\right)\end{aligned}$$

[1 mark]
where the time variable in the last line is t (as opposed to t' inside the integral). Set this to zero
[1 mark]
to find

$$\ddot{x} = -\omega^2 x.$$

A.3 Solve for the classical trajectory $x(t)$ assuming $x(t=0)=0$ and $x(t=T)=X$.

[3 marks]

The general solution to the equation in A.2 is

$$x(t) = A\exp(i\omega t) + B\exp(-i\omega t).$$

[1 mark]
We require that

$$x(0) = 0$$

giving

$$x(t) = A\sin(\omega t)$$

[1 mark]
and

$$x(T) = X$$

giving

$$x(t) = X\frac{\sin(\omega t)}{\sin(\omega T)}. \tag{5}$$

[1 mark]

Section B: bringing together ideas from across the course.

B.1 Explain the method of stationary phase.

[2 marks]

The propagator is a functional integral over all possible trajectories, weighted by a phase $\exp(iS/\hbar)$.
[1 mark]
The method of stationary phase acknowledges that the biggest contribution to the integral comes from those paths near the classical paths, for which the variation of S is zero.
[1 mark]

For a propagator

$$K(x,t;x_0,t_0) = \int \mathcal{D}x\exp(iS[x]/\hbar) \tag{6}$$

the method of stationary phase gives

$$K(x, t; x_0, t_0) \approx \mathcal{Z}^{-1} \exp\left(iS[x_c]/\hbar\right)$$

where $x_c(t)$ is a classical trajectory.

B.2 Find an expression for the normalisation $\mathcal{Z}^{-1}$ in terms of a Gaussian functional integral (which you do not need to evaluate).

[3 marks]

Expand the action around the classical trajectory using a Taylor series:

$$S[x] = S[x_c] + (x - x_c) \left.\frac{\delta S}{\delta x}\right|_{x=x_c} + \frac{1}{2}(x - x_c)^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c} + ...$$

[1 mark]
and note that, by definition, the second term vanishes for classical trajectories, so

$$S[x] \approx S[x_c] + \frac{1}{2}(x - x_c)^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}$$

[1 mark]
therefore

$$K(X, T; 0, 0) \approx \exp\left(iS[x_c]/\hbar\right) \int \mathcal{D}x \exp\left(\frac{i}{2\hbar}(x - x_c)^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}\right).$$

A change of variables will remove the x_c term to give the simple Gaussian functional integral

$$K(X, T; 0, 0) \approx \exp\left(iS[x_c]/\hbar\right) \int \mathcal{D}x \exp\left(\frac{i}{2\hbar}x^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}\right).$$

The normalisation is therefore

$$\mathcal{Z}^{-1} = \int \mathcal{D}x \exp\left(\frac{i}{2\hbar}x^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}\right).$$

[1 mark]

B.3 Using the method of stationary phase and your answer to **A.3**, find an approximate expression for the propagator of the harmonic oscillator. You do not need to evaluate the normalisation $\mathcal{Z}$.

[4 marks]

$$\begin{aligned}
S[x_c] &= \frac{1}{2}m \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right) \\
&= \frac{m\omega^2 X^2}{\sin^2(\omega T)} \int_0^T \mathrm{d}t' \sin^2(\omega t') \\
&= \frac{m\omega^2 X^2}{\sin^2(\omega T)} \int_0^T \mathrm{d}t' \frac{1 - \cos(2\omega t')}{2} \\
&= \frac{1}{2}\frac{m\omega^2 X^2}{\sin^2(\omega T)} \left[T - \frac{\sin(2\omega T)}{2\omega}\right]
\end{aligned}$$

[4 marks]
So finally

$$K\left(X,T;0,0\right)=\mathcal{Z}^{-1}\exp\left(i\frac{m\omega^2X^2}{2\hbar\sin^2\left(\omega T\right)}\left[T-\frac{\sin\left(2\omega T\right)}{2\omega}\right]\right).$$

[1 mark]

Section C: more challenging.

Now consider the harmonic oscillator forced with a time-independent force F, for which the action is

$$S\left[x\right]=\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-Fx\right)\mathrm{d}t'. \tag{7}$$

C.1 Show that the propagator is

$$K_{forced}\left(X,T;0,0\right)=\exp\left(i\epsilon T/\hbar\right)K\left(X,T;0,0\right)$$

where ϵ is an energy you should find.

[5 marks]

We can complete the square on the potential terms:
[1 mark]

$$K_{forced}\left(X,T;0,0\right)=\int\mathcal{D}x\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2\left(x^2-\frac{2}{m\omega^2}Fx\right)\right)\mathrm{d}t'/\hbar\right)$$
$$K_{forced}\left(X,T;0,0\right)=\int\mathcal{D}x\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2\left(x-\frac{1}{m\omega^2}F\right)^2+\frac{1}{2m\omega^2}F^2\right)\mathrm{d}t'/\hbar\right)$$

[1 mark]
and the remaining F^2 term pulls out of the functional integral:

$$K_{forced}\left(X,T;0,0\right)=\exp\left(i\int_0^T\frac{1}{2m\omega^2}F^2\mathrm{d}t'/\hbar\right)\int\mathcal{D}x\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2\left(x-\frac{1}{m\omega^2}F\right)^2\right)\mathrm{d}t'/\hbar\right)$$

[1 mark].
Finally notice that we can change variables in the functional integral to

$$y\left(t\right)=x\left(t\right)-\frac{1}{m\omega^2}F$$
$$\dot{y}\left(t\right)=\dot{x}\left(t\right)$$

giving

$$K_{forced}\left(X,T;0,0\right)=\exp\left(i\int_0^T\frac{1}{2m\omega^2}F^2\mathrm{d}t'/\hbar\right)\int\mathcal{D}y\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2\right)\mathrm{d}t'/\hbar\right)$$
$$=\exp\left(i\int_0^T\frac{1}{2m\omega^2}F^2\mathrm{d}t'/\hbar\right)K\left(X,T;0,0\right)$$

[1 mark]
and finally the integral is trivial as F is constant:

$$K_{forced}\left(X,T;0,0\right)=\exp\left(i\frac{F^2T}{2m\hbar\omega^2}\right)K\left(X,T;0,0\right)$$

giving

$$\epsilon=\frac{F^2}{2m\omega^2}$$

[1 mark]

2 Semiclassics

In this question we will apply the WKB approximation to the harmonic oscillator.

Section A: mostly bookwork.

A.1 Derive the WKB approximation

$$\psi(x) = \frac{A_+}{\sqrt{p(x)}} \exp\left(i \int p(x)\, \mathrm{d}x/\hbar\right) - \frac{A_-}{\sqrt{p(x)}} \exp\left(-i \int p(x)\, \mathrm{d}x/\hbar\right) \tag{8}$$

[7 marks]

This is straight from the notes, so I will not write out the solution. But you should work through it at least once!

The energy of the classical harmonic oscillator is

$$E = \frac{p(x)^2}{2m} + \frac{1}{2} m\omega^2 x^2. \tag{9}$$

A.2 Identify the two classical turning points $x_\pm$ of the harmonic oscillator potential.

[3 marks]

The classical turning points occur when $E = V(x) = \frac{1}{2} m\omega^2 x^2$.
[1 mark]
Therefore

$$x_\pm = \pm\sqrt{\frac{2E}{m\omega^2}}$$

[2 marks]

Section B: bringing together ideas from across the course.

B.1 The classical turning points of the harmonic oscillator are 'soft'. Explain how this leads to the Bohr Sommerfeld quantization condition

$$\oint p\mathrm{d}x = 2\pi\hbar\left(n + \frac{1}{2}\right). \tag{10}$$

[2 marks]

For soft turning points, we need to add a $\pi/2$ phase shift for each classical turning point in the cycle. I don't have a good analogy for this; it comes from Morse theory, and is called the Maslov index. It is complicated, and its consideration led to a major result in algebraic geometry / string theory (Witten's conjecture). But it's worth remembering. Hard boundaries give a π phase shift, which matches that of a classical wave reflected at a hard boundary; whether there is a deeper connection I'm not sure!
[1 mark] for something sensible.
In any case, the soft turning points lead to the extra factor of $1/2$, as there are two soft boundaries per cycle.
[1 mark]

B.2 Using the Bohr Sommerfeld quantization condition, find the WKB approximation for energy of the quantum particle in the harmonic oscillator.

[6 marks]

$$p(x) = \sqrt{2mE - m^2\omega^2 x^2}$$

[1 mark]
where the + sign is chosen by convention. In the Bohr Sommerfeld condition, the integral is twice the distance between the classical turning points.
[1 mark]
Therefore

$$\begin{aligned}\oint p(x)\,\mathrm{d}x &= 2\int_{x_-}^{x_+} \sqrt{2mE - m^2\omega^2 x^2}\mathrm{d}x \\ &= 2\sqrt{2mE}\int_{x_-}^{x_+} \sqrt{1 - \frac{m\omega^2}{2E}x^2}\mathrm{d}x\end{aligned}$$

now change variables using

$$\begin{aligned}\sin(\theta) &= \sqrt{\frac{m\omega^2}{2E}}x \\ \cos(\theta)\,\mathrm{d}\theta &= \sqrt{\frac{m\omega^2}{2E}}\mathrm{d}x\end{aligned}$$

[1 mark]
and the integration limits are now

$$\begin{aligned}x_\pm &= \pm\sqrt{\frac{2E}{m\omega^2}} \\ &\downarrow \\ \sin(\theta) &= \pm 1 \\ \theta &= \pm\pi/2\end{aligned}$$

[1 mark]

$$\begin{aligned}\oint p(x)\,\mathrm{d}x &= 2\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}}\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\,\mathrm{d}\theta \\ &= \frac{2E}{\omega}\int_{-\pi/2}^{\pi/2} (1 + \cos(2\theta))\,\mathrm{d}\theta \\ &= \frac{2E}{\omega}\pi\end{aligned}$$

[1 mark]
Therefore

$$\frac{2E}{\omega}\pi = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

and

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

[1 mark]

B.3 Explain for which energies the WKB approximation to the wavefunction will be the most accurate.

[2 marks]

The WKB approximation is semi-classical, and the correspondence principle states that quantum mechanics approaches classical mechanics in the limit of large quantum number. Hence, it is most accurate for large energies. Any statement to a similar effect receives full marks.

Section C: more challenging.

C.1 Explain with the aid of a diagram the possible trajectories undertaken by the quantum particle through phase space (x, p), including the areas enclosed.

[5 marks]

$$\hbar\omega\left(n+\frac{1}{2}\right)=\frac{p\left(x\right)^2}{2m}+\frac{1}{2}m\omega^2x^2$$

This set of equations describes concentric ellipses centred on $(p, x) = \mathbf{0}$.
[1 mark]
It will be easiest to plot using scaled axes:

$$n+\frac{1}{2}=p'^2+x'^2$$

$$p'=p/\sqrt{2m\hbar\omega}$$
$$x'=x\sqrt{\frac{m\omega}{2\hbar}}$$

in which case the classical trajectories are simply concentric circles of radius $\sqrt{n+\frac{1}{2}}$.
[1 mark]
The Bohr Sommerfeld condition tells us that the area enclosed in (x, p) by ellipse n must be $(n + 1/2)\,\hbar\omega$, or area $\pi\left(n+\frac{1}{2}\right)$ in (x', p') which happens to be equal to the full quantum solution for the harmonic oscillator.
[1 mark]
Finally,
[2 marks]
For a decent picture summarising the above. Rescaling is not important provided the general shapes are correct, or axis intercepts are clearly labelled.

Semiclassics – solutions

In this question we will apply the WKB approximation to the harmonic oscillator.

Section A: mostly bookwork.

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$$\psi(x) = \frac{A_+}{\sqrt{p(x)}} \exp\left(i \int p(x)\, \mathrm{d}x/\hbar\right) - \frac{A_-}{\sqrt{p(x)}} \exp\left(-i \int p(x)\, \mathrm{d}x/\hbar\right) \tag{11}$$

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This is straight from the notes, so I will not write out the solution. But you should work through it at least once!

The energy of the classical harmonic oscillator is

$$E = \frac{p(x)^2}{2m} + \frac{1}{2} m\omega^2 x^2. \tag{12}$$

A.2 Identify the two classical turning points $x_\pm$ of the harmonic oscillator potential.

[3 marks]

The classical turning points occur when $E = V(x) = \frac{1}{2} m\omega^2 x^2$.
[1 mark]
Therefore

$$x_\pm = \pm\sqrt{\frac{2E}{m\omega^2}}$$

[2 marks]

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$$\oint p\mathrm{d}x = 2\pi\hbar\left(n + \frac{1}{2}\right). \tag{13}$$

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where the + sign is chosen by convention. In the Bohr Sommerfeld condition, the integral is twice the distance between the classical turning points.
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now change variables using

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