Advanced Quantum Physics: Problem Set 4

This problem set is due of Friday of Week 5.

Problem set marks

- Satisfactory completion of each of the four problem sets will receive 5% of the final mark for the course.
- ‘Satisfactory completion’ is defined as follows. You must *either* (i) make a written attempt to achieve each of the 25 marks on Question 2, *or* (ii) for any marks on Q2 you are not sure how to achieve, write a short paragraph explaining where you are stuck, listing the sections of three separate resources you have consulted.
- **Note**: the marking criteria for problem sets therefore do not require you to get any answers correct to receive full marks! In the exam you will need to get correct answers to receive the marks.
- You are expected to spend 10 hours per week on this course, including lectures and problems classes. That leaves you with around 6 hours per week to work on problem sets. The marking scheme is designed to allow you flexibility to direct your own learning: you may wish to use some of your time to revise relevant parts of other courses.

Format of problem sets

- In the exam you will need to answer four 25-mark questions in 2 hours. You will not have access to notes.
- Question 1 (worked example) and Question 2 are designed to closely mimic an exam question.
- Try to work through the worked example by yourself before looking at the answers.

1 Time Evolution (worked example)

In this question we will work through another example of first order time dependent scattering in the Interaction picture.

Section A: mostly bookwork.

In the Interaction picture, the Hamiltonian is assumed to take the form

$$\hat{H} = \hat{H}_0 + \hat{V}(t) \tag{1}$$

where $\hat{H}_0$ is time independent. Time evolution takes the form

$$|\psi_I(t_f)\rangle = |\psi_I(t_i)\rangle - \frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t' \hat{V}_I(t')\,|\psi_I(t')\rangle \tag{2}$$

where

$$\hat{V}_I(t) \triangleq \exp\left(i\hat{H}_0 t/\hbar\right)\hat{V}(t)\exp\left(-i\hat{H}_0 t/\hbar\right). \tag{3}$$

A.1 Explain why expectation values must be independent of the picture in which one works.

[3 marks]

A.2 The amplitude for an initial state $|i(t_i)\rangle$ to scatter to a final state $|f(t_f)\rangle$, to first order in time-dependent perturbation theory and working in the interaction picture, is

$$\langle f_I(t_f)\,|i_I(t_i)\rangle = \langle f_I(t_f)\,|i_I(t_f)\rangle + \frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t' \langle f_I(t_f)\,|\hat{V}_I(t')\,|i_I(t_f)\rangle. \tag{4}$$

Show that this can be written as

$$\langle f_I(t_f)\,|i_I(t_i)\rangle = \langle f|i\rangle + \frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t' \langle f_I(t_f)\,|\exp\left(i\hat{H}_0 t'/\hbar\right)\hat{V}(t')\exp\left(-i\hat{H}_0 t'/\hbar\right)|i_I(t_f)\rangle.$$

[3 marks]

A.3 The interaction picture is typically applied when the eigenbasis of $\hat{H}_0$ is known. Consider the simplest case, where ingoing and outgoing states are eigenstates of $\hat{H}_0$:

$$|i_{I,n}\rangle = |f_{I,n}\rangle = |n_I(t)\rangle \tag{5}$$

$$\hat{H}_0|n_S\rangle = E_n|n_S\rangle \tag{6}$$

and

$$\begin{aligned}|n_I(t)\rangle &= \exp\left(i\hat{H}_0 t/\hbar\right)|n_S(t)\rangle \\ &= |n_S(0)\rangle \\ &\triangleq |n_S\rangle.\end{aligned}$$

Under these circumstances, show that

$$\langle m_I(t_f)\,|n_I(t_i)\rangle = \delta_{mn} + \frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t' \exp\left(i\left(E_m - E_n\right)t'/\hbar\right)\langle m_S|\hat{V}(t')\,|n_S\rangle.$$

[4 marks]

Section B: bringing together ideas from across the course.

From now on we will assume that $\hat{H}_0$ takes the form of an infinite square well. In the position basis,

$$\hat{H}_0 \phi_n(x) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0(x)\right)\phi_n(x) \tag{7}$$

where

$$V_0(x) = \begin{cases} 0, & 0 \le x \le L \\ \infty, & \text{otherwise.} \end{cases} \tag{8}$$

The energy eigenstates and eigenvalues are

$$\langle x|n\rangle = \phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$
$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}.$$

The time-dependent perturbation takes the form

$$\hat{V}(t) = -eEg(t)\,\hat{x} \tag{9}$$

where $g(t)$ is some function of time that turns on only during a finite interval.

B.1 Working to first order, show that the amplitude for an initial state $\phi_n(x, t=-\infty)$ to scatter to a final state $\phi_m(x, t=+\infty)$ is

$$\langle m_I(\infty)|n_I(-\infty)\rangle = \delta_{mn} - \frac{2ieE}{L\hbar}\int_{-\infty}^{\infty} \mathrm{d}t' g(t')\exp\left(i\frac{\hbar\pi^2(m^2-n^2)t'}{2mL^2}\right)\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

[5 marks]

B.2 Show that any scattering of the form

$$\phi_{n\,\text{odd}} \to \phi_{m\,\text{odd}}$$

or

$$\phi_{n\,\text{even}} \to \phi_{m\,\text{even}}$$

is zero, where $n \neq m$.

[5 marks]

Section C: more challenging.

C.1 Using the definition of the Fourier transform

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)\exp(i\omega t)\,\mathrm{d}t \tag{10}$$

find an expression for the probability to scatter from $\phi_{n\,\text{even}} \to \phi_{m\,\text{odd}}$, or $\phi_{n\,\text{odd}} \to \phi_{m\,\text{even}}$.

[5 marks]

Time Evolution (worked example solutions)

In this question we will work through another example of first order time dependent scattering in the Interaction picture.

Section A: mostly bookwork.

In the Interaction picture, the Hamiltonian is assumed to take the form

$$\hat{H} = \hat{H}_0 + \hat{V}(t) \tag{11}$$

where $\hat{H}_0$ is time independent. Time evolution takes the form

$$|\psi_I(t_f)\rangle = |\psi_I(t_i)\rangle - \frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t' \hat{V}_I(t')\,|\psi_I(t')\rangle \tag{12}$$

where

$$\hat{V}_I(t) \triangleq \exp\left(i\hat{H}_0 t/\hbar\right)\hat{V}(t)\exp\left(-i\hat{H}_0 t/\hbar\right). \tag{13}$$

A.1 Explain why expectation values must be independent of the picture in which one works.

[3 marks]

The different pictures are simply mathematical descriptions. Physically, expectation values are measurable quantities. They cannot, therefore, depend on the mathematical description one uses for them. Such an answer would obtain full marks.
Alternatively we can give a mathematical answer. Mathematically, we could compare to the Schroedinger picture, say:

$$\begin{aligned} |\psi_I(t)\rangle &= \exp\left(i\hat{H}_0 t/\hbar\right)|\psi_S(t)\rangle \\ \hat{A}_I(t) &= \exp\left(i\hat{H}_0 t/\hbar\right)\hat{A}_S \exp\left(-i\hat{H}_0 t/\hbar\right) \end{aligned}$$

so an arbitrary expectation value

$$\langle\varphi|\hat{A}|\psi\rangle = \langle\varphi_S(t)|\hat{A}_S|\psi_S(t)\rangle = \langle\varphi_I(t)|\hat{A}_I(t)|\psi_I(t)\rangle.$$

A.2 The amplitude for an initial state $|i(t_i)\rangle$ to scatter to a final state $|f(t_f)\rangle$, to first order in time-dependent perturbation theory and working in the interaction picture, is

$$\langle f_I(t_f)|i_I(t_i)\rangle = \langle f_I(t_f)|i_I(t_f)\rangle + \frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t' \langle f_I(t_f)|\hat{V}_I(t')|i_I(t_f)\rangle. \tag{14}$$

Show that this can be written as

$$\langle f_I(t_f)|i_I(t_i)\rangle = \langle f|i\rangle + \frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t' \langle f_I(t_f)|\exp\left(i\hat{H}_0 t'/\hbar\right)\hat{V}(t')\exp\left(-i\hat{H}_0 t'/\hbar\right)|i_I(t_f)\rangle.$$

[3 marks]

The first term has both ingoing and outgoing states evaluated at the same time. Therefore we can drop the reference to the picture.
[1 mark]
In the last term just rewrite the potential in the Schroedinger picture.
[2 mark]
A.3 The interaction picture is typically applied when the eigenbasis of $\hat{H}_0$ is known. Consider the simplest case, where ingoing and outgoing states are eigenstates of $\hat{H}_0$:

$$|i_{I,n}\rangle = |f_{I,n}\rangle = |n_I(t)\rangle \tag{15}$$
$$\hat{H}_0|n_S\rangle = E_n|n_S\rangle \tag{16}$$

and

$$\begin{aligned}|n_I(t)\rangle &= \exp\left(i\hat{H}_0 t/\hbar\right)|n_S(t)\rangle \\ &= |n_S(0)\rangle \\ &\triangleq |n_S\rangle.\end{aligned}$$

Under these circumstances, show that

$$\langle m_I(t_f)|n_I(t_i)\rangle = \delta_{mn} + \frac{i}{\hbar}\mathrm{d}t' \exp\left(i\left(E_m - E_n\right)t'/\hbar\right)\langle m_S|\hat{V}(t')|n_S\rangle$$

[4 marks]

$$\langle m_I(t_f)|n_I(t_i)\rangle = \langle m|n\rangle + \frac{i}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t'\langle m_I(t_f)|\exp\left(i\hat{H}_0 t'/\hbar\right)\hat{V}(t')\exp\left(-i\hat{H}_0 t'/\hbar\right)|n_I(t_f)\rangle$$

Given the ingoing and outgoing states are eigenstates of the same Hamiltonian, the are assumed to be part of an orthonormal basis. Therefore

$$\langle m|n\rangle = \delta_{nm}$$

where δ_{mn} is zero if the states are different, 1 if they are the same state (this is the definition of the Kronecker delta, but you needn't remember its name!).
[2 marks]
Then

$$\begin{aligned}\langle m_I(t_f)|n_I(t_i)\rangle &= \delta_{mn} + \frac{i}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t'\langle m_I(t_f)|\exp\left(i\hat{H}_0 t'/\hbar\right)\hat{V}(t')\exp\left(-i\hat{H}_0 t'/\hbar\right)|n_I(t_f)\rangle \\ &= \delta_{mn} + \frac{i}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t'\langle m_S|\exp\left(i\hat{H}_0 t'/\hbar\right)\hat{V}(t')\exp\left(-i\hat{H}_0 t'/\hbar\right)|n_S\rangle \\ &= \delta_{mn} + \frac{i}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t'\langle m_S|\exp\left(iE_m t'/\hbar\right)\hat{V}(t')\exp\left(-iE_n t'/\hbar\right)|n_S\rangle \\ &= \delta_{mn} + \frac{i}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t'\exp\left(i\left(E_m - E_n\right)t'/\hbar\right)\langle m_S|\hat{V}(t')|n_S\rangle\end{aligned}$$

[2 marks]

Section B: bringing together ideas from across the course.

From now on we will assume that $\hat{H}_0$ takes the form of an infinite square well. In the position basis,

$$\hat{H}_0\phi_n(x) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0(x)\right)\phi_n(x) \tag{17}$$

where

$$V_0(x) = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & \text{otherwise} \end{cases}. \tag{18}$$

The energy eigenstates and eigenvalues are

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$
$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}.$$

The time-dependent perturbation takes the form

$$\hat{V}(t) = -eEg(t)\hat{x} \tag{19}$$

where $g(t)$ is some function of time that turns on only during a finite interval of time.

B.1 Working to first order, show that the amplitude for an initial state $\phi_n(x, t = -\infty)$ to scatter to a final state $\phi_m(x, t = +\infty)$ is

$$\langle m_I(\infty)|n_I(-\infty)\rangle = \delta_{mn} - \frac{2ieE}{L\hbar}\int_{-\infty}^{\infty} \mathrm{d}t' g(t')\exp\left(i\frac{\hbar\pi^2(m^2-n^2)t'}{2mL^2}\right)\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

[5 marks]

$$\langle m_I(\infty)|n_I(-\infty)\rangle = \delta_{mn} + \frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t'\exp\left(i(E_m - E_n)t'/\hbar\right)\langle m_S|\hat{V}(t')|n_S\rangle$$

[1 mark]
Next,

$$E_m - E_n = \frac{\hbar^2\pi^2(m^2-n^2)}{2mL^2}$$

[1 mark]
giving

$$\langle m_I(\infty)|n_I(-\infty)\rangle = \delta_{mn} + \frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t'\exp\left(i(E_m - E_n)t'/\hbar\right)\langle m_S|\hat{V}(t')|n_S\rangle$$

giving

$$\langle m_I(\infty)|n_I(-\infty)\rangle = \delta_{mn} + \frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t'\exp\left(i\frac{\hbar\pi^2(m^2-n^2)t'}{2mL^2}\right)\langle m_S|\hat{V}(t')|n_S\rangle$$

[1 mark]
(note the single power of $\hbar$). Insert the potential:

$$\langle m_I(\infty)|n_I(-\infty)\rangle = \delta_{mn} - \frac{ieE}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t' g(t') \exp\left(i\frac{\hbar\pi^2(m^2-n^2)t'}{2mL^2}\right)\langle m_S|\hat{x}|n_S\rangle$$

[1 mark]
Finally, in the position basis

$$\begin{aligned}\langle m_S|\hat{x}|n_S\rangle &= \int_0^L \phi_m^*(x)\, x\phi_n(x)\,\mathrm{d}x \\ &= \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x\end{aligned}$$

[1 mark] giving

$$\langle m_I(\infty)|n_I(-\infty)\rangle = \delta_{mn} - \frac{2ieE}{L\hbar}\int_{-\infty}^{\infty} \mathrm{d}t' g(t') \exp\left(i\frac{\hbar\pi^2(m^2-n^2)t'}{2mL^2}\right)\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

B.2 Show that any scattering of the form

$$\phi_{n\,\mathrm{odd}} \to \phi_{m\,\mathrm{odd}}$$

or

$$\phi_{n\,\mathrm{even}} \to \phi_{m\,\mathrm{even}}$$

is zero, where $n \neq m$.

[5 marks]

We must solve

$$\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

Hopefully we're very familiar with this calculation from 2nd year!

$$\begin{aligned}\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\ \cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)\end{aligned}$$

[1 mark]
so

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left(\cos\left((n-m)\frac{\pi x}{L}\right) - \cos\left((n+m)\frac{\pi x}{L}\right)\right)$$

$$\begin{aligned}\int_0^L \phi_m^*(x)\, x\phi_n(x)\,\mathrm{d}x &= \frac{1}{2}\int_0^L x\left(\cos\left((n-m)\frac{\pi x}{L}\right) - \cos\left((n+m)\frac{\pi x}{L}\right)\right)\mathrm{d}x \\ &= \frac{L}{2\pi}\left[x\left(\frac{\sin\left((n-m)\frac{\pi x}{L}\right)}{n-m} - \frac{\sin\left((n+m)\frac{\pi x}{L}\right)}{n+m}\right)\right]_0^L \\ &\quad - \frac{L}{2\pi}\int_0^L \left(\frac{\sin\left((n-m)\frac{\pi x}{L}\right)}{n-m} - \frac{\sin\left((n+m)\frac{\pi x}{L}\right)}{n+m}\right)\mathrm{d}x\end{aligned}$$

[2 marks]
The boundary term vanishes.
[1 mark]

$$\int_0^L \phi_m^*(x)\, x \phi_n(x)\, \mathrm{d}x = \frac{L^2}{2\pi^2}\left[\frac{\cos\left((n-m)\frac{\pi x}{L}\right)}{(n-m)^2} - \frac{\cos\left((n+m)\frac{\pi x}{L}\right)}{(n+m)^2}\right]_0^L$$
$$= \frac{L^2}{2\pi^2}\left[\frac{\cos((n-m)\pi)-1}{(n-m)^2} - \frac{\cos((n+m)\pi)-1}{(n+m)^2}\right]$$

If n and m are both odd or both even, the numerators vanish. Since $n \neq m$ the denominators do not, and the result is zero.
[1 mark]
Intuitively, this is because the potential (in the position basis) is an odd function, and the product of eigenstates is an even function. In principle a clear enough statement of this fact would receive full marks, without the working, but it would have to be very clear (e.g. the relevance of odd/even needs to be explained, given the fact that the well goes from 0 to L).

Section C: more challenging.

C.1 Using the definition of the Fourier transform

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t) \exp(i\omega t)\, \mathrm{d}t \tag{20}$$

find an expression for the probability to scatter from $\phi_{n\,\text{even}} \to \phi_{m\,\text{odd}}$, or $\phi_{n\,\text{odd}} \to \phi_{m\,\text{even}}$.

[5 marks]

In this case,

$$\int_0^L \phi_m^*(x)\, x \phi_n(x)\, \mathrm{d}x = -2\frac{L^2}{2\pi^2}\left[\frac{1}{(n-m)^2} - \frac{1}{(n+m)^2}\right]$$
$$= -\frac{4nmL^2}{(n^2-m^2)^2\pi^2}$$

[2 marks]
so

$$\langle m_I(\infty) | n_I(-\infty)\rangle = \frac{8ieEnmL}{\hbar(n^2-m^2)^2\pi^2}\int_{-\infty}^{\infty} \mathrm{d}t' g(t') \exp\left(i\frac{\hbar\pi^2(m^2-n^2)t'}{2mL^2}\right)$$

[1 mark]
noting that $\delta_{nm} = 0$ since $n \neq m$. The amplitude is therefore

$$\langle m_I(\infty) | n_I(-\infty)\rangle = \frac{8ieEnmL}{\hbar(n^2-m^2)^2\pi^2}\tilde{g}\left(\frac{\hbar\pi^2(m^2-n^2)}{2mL^2}\right)$$

[1 mark]
and the probability

$$\mathrm{P}(n \to m) = \frac{8^2e^2E^2n^2m^2L^2}{\hbar^2(n^2-m^2)^4\pi^4}\left|\tilde{g}\left(\frac{\hbar\pi^2(m^2-n^2)}{2mL^2}\right)\right|^2.$$

[1 mark]

2 Time Evolution (Question to answer)

Section A: mostly bookwork.

A.1 Explain the Schrödinger and Heisenberg pictures, including their inter-relation.

[6 marks]

A.2 In the Schrödinger picture, time evolution is governed by the TDSE:

$$i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle = \hat{H}|\psi_S(t)\rangle. \tag{21}$$

State the equation governing time evolution in the Heisenberg picture.

[2 marks]

A.3 In the Schrödinger picture, show that

$$|\psi_S(t)\rangle = \exp\left(-i\hat{H}t/\hbar\right)|\psi_S(0)\rangle \tag{22}$$

solves the TDSE.

[2 marks]

Section B: bringing together ideas from across the course.

In the interaction picture,

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) \tag{23}$$

where $\hat{H}_0$ has no explicit time dependence, but there is an explicit time dependence of the potential. We also have the following general relation for the time dependence of operators in the Interaction picture:

$$\hat{A}_I(t) = \exp\left(i\hat{H}_0 t/\hbar\right)\hat{A}(t)\exp\left(-i\hat{H}_0 t/\hbar\right). \tag{24}$$

The relation to the Schrödinger picture comes from

$$|\psi_I(t)\rangle = \exp\left(i\hat{H}_0 t/\hbar\right)|\psi_S(t)\rangle. \tag{25}$$

B.1 Show that the time evolution in the interaction picture is described by the equation

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \hat{V}_I(t)|\psi_I(t)\rangle. \tag{26}$$

Hint: use Eq.25 and the TDSE, Eq.21.

[10 marks]

Section C: more challenging.

C.1 Derive an equation for the time dependence of operators in the Interaction picture, similar to your answer to A.2.

[5 marks]

Time Evolution (solutions)

Section A: mostly bookwork.

A.1 Explain the Schrödinger and Heisenberg pictures, including their inter-relation.

[6 marks]

Schrödinger: time dependent states, time independent operators.
[1 mark]

$$
\begin{aligned}
|\psi_S(t)\rangle &= \exp\left(-i\hat{H}t/\hbar\right)|\psi_S(0)\rangle \\
\hat{A}_S(t) &= \hat{A}_S(0)
\end{aligned}
$$

[1 mark]
Heisenberg: time independent states, time dependent operators:
[1 mark]

$$
\begin{aligned}
|\psi_H(t)\rangle &= |\psi_H(0)\rangle \\
\hat{A}_H(t) &= \exp\left(i\hat{H}t/\hbar\right)\hat{A}_H(t)\exp\left(-i\hat{H}t/\hbar\right)
\end{aligned}
$$

[1 mark]
Interaction: define

$$\hat{H} = \hat{H}_0 + \hat{V}(t).$$

Inter-relation:

$$
\begin{aligned}
|\psi_S(t)\rangle &= \exp(i\phi)\exp\left(-i\hat{H}t/\hbar\right)|\psi_H\rangle \\
\hat{A}_H(t) &= \exp\left(-i\hat{H}t/\hbar\right)\hat{A}_S\exp\left(i\hat{H}t/\hbar\right)
\end{aligned}
$$

[2 marks]

A.2 In the Schrödinger picture, time evolution is governed by the TDSE:

$$i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle = \hat{H}|\psi_S(t)\rangle. \tag{27}$$

State the equation governing time evolution in the Heisenberg picture.

[2 marks]

We saw this in the problems class:

$$\frac{\mathrm{d}\hat{A}_H(t)}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{A}_H(t),\hat{H}\right] + \left(\frac{\partial\hat{A}_S(t)}{\partial t}\right)_H$$

where the notation on the final term indicates that a partial derivative is taken with respect to time in the Schroedinger picture, then the result is converted to the Heisenberg picture. Equivalently,

$$\frac{\mathrm{d}\hat{A}_H(t)}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{A}_H(t),\hat{H}\right] + \exp\left(i\hat{H}t/\hbar\right)\frac{\partial\hat{A}_S(t)}{\partial t}\exp\left(-i\hat{H}t/\hbar\right).$$

A.3 In the Schrödinger picture, show that

$$|\psi_S(t)\rangle = \exp\left(-i\hat{H}t/\hbar\right)|\psi_S(0)\rangle \tag{28}$$

solves the TDSE.

[2 marks]

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle &= i\hbar\frac{\partial}{\partial t}\exp\left(-i\hat{H}t/\hbar\right)|\psi_S(0)\rangle \\
&= i\hbar\left(-i\hat{H}/\hbar\right)\exp\left(-i\hat{H}t/\hbar\right)|\psi_S(0)\rangle \\
&= \hat{H}\exp\left(-i\hat{H}t/\hbar\right)|\psi_S(0)\rangle \\
&= |\psi_S(t)\rangle
\end{aligned}$$

Section B: bringing together ideas from across the course.

In the interaction picture,

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) \tag{29}$$

where $\hat{H}_0$ has no explicit time dependence, but there is an explicit time dependence of the potential. We also have the following general relation for the time dependence of operators in the Interaction picture:

$$\hat{A}_I(t) = \exp\left(i\hat{H}_0 t/\hbar\right)\hat{A}(t)\exp\left(-i\hat{H}_0 t/\hbar\right). \tag{30}$$

The relation to the Schrödinger picture comes from

$$|\psi_I(t)\rangle = \exp\left(i\hat{H}_0 t/\hbar\right)|\psi_S(t)\rangle. \tag{31}$$

B.1 Show that the time evolution in the interaction picture is described by the equation

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \hat{V}_I(t)|\psi_I(t)\rangle. \tag{32}$$

Hint: use Eq.31 and the TDSE, Eq.27.

[10 marks]

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle &= i\hbar\frac{\partial}{\partial t}\left(\exp\left(i\hat{H}_0 t/\hbar\right)|\psi_s(t)\rangle\right) \\
&= -\hat{H}_0\exp\left(i\hat{H}_0 t/\hbar\right)|\psi_s(t)\rangle + \exp\left(i\hat{H}_0 t/\hbar\right)i\hbar\frac{\partial}{\partial t}|\psi_s(t)\rangle
\end{aligned}$$

[1 mark]

Now commute the $\hat{H}_0$ through the $\exp\left(i\hat{H}_0 t/\hbar\right)$ in the first term

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \exp\left(i\hat{H}_0 t/\hbar\right)\left(-\hat{H}_0\right)|\psi_s(t)\rangle + \exp\left(i\hat{H}_0 t/\hbar\right)i\hbar\frac{\partial}{\partial t}|\psi_s(t)\rangle$$

[1 mark]
and use the TDSE to rewrite the second term:

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \exp\left(i\hat{H}_0 t/\hbar\right)\left(-\hat{H}_0\right)|\psi_s(t)\rangle + \exp\left(i\hat{H}_0 t/\hbar\right)\hat{H}|\psi_s(t)\rangle$$

[1 mark]
and regroup:

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \exp\left(i\hat{H}_0 t/\hbar\right)\left(\hat{H} - \hat{H}_0\right)|\psi_s(t)\rangle$$

[1 mark]
giving

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \exp\left(i\hat{H}_0 t/\hbar\right)\hat{V}(t)|\psi_s(t)\rangle$$

[1 mark].
Now insert an identity of the form

$$\hat{\mathbb{I}} = \exp\left(-i\hat{H}_0 t/\hbar\right)\exp\left(i\hat{H}_0 t/\hbar\right)$$

[1 mark]
to give

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \exp\left(i\hat{H}_0 t/\hbar\right)\hat{V}(t)\exp\left(-i\hat{H}_0 t/\hbar\right)\exp\left(i\hat{H}_0 t/\hbar\right)|\psi_s(t)\rangle$$

[1 mark].
Finally, note that

$$\exp\left(i\hat{H}_0 t/\hbar\right)\hat{V}(t)\exp\left(-i\hat{H}_0 t/\hbar\right) = \hat{V}_I(t)$$

[1 mark]
and

$$\exp\left(i\hat{H}_0 t/\hbar\right)|\psi_s(t)\rangle = |\psi_I(t)\rangle$$

[1 mark]
to give

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \hat{V}_I(t)|\psi_I(t)\rangle$$

as required.
[1 mark]

Section C: more challenging.

C.1 Derive an equation for the time dependence of operators in the Interaction picture, similar to your answer to A.2.

[5 marks]

Start from

$$\hat{A}_I(t) = \exp\left(i\hat{H}_0 t/\hbar\right)\hat{A}(t)\exp\left(-i\hat{H}_0 t/\hbar\right)$$

and act $i\hbar\partial_t$ as before:

$$
\begin{aligned}
i\hbar\frac{\partial \hat{A}_I(t)}{\partial t} &= -\hat{H}_0 \exp\left(i\hat{H}_0 t/\hbar\right)\hat{A}(t)\exp\left(-i\hat{H}_0 t/\hbar\right) \\
&\quad + \exp\left(i\hat{H}_0 t/\hbar\right)\left(i\hbar\frac{\partial \hat{A}(t)}{\partial t}\right)\exp\left(-i\hat{H}_0 t/\hbar\right) \\
&\quad + \exp\left(i\hat{H}_0 t/\hbar\right)\hat{A}(t)\exp\left(-i\hat{H}_0 t/\hbar\right)\hat{H}_0
\end{aligned}
$$

applying the chain rule in the second line, to each of the three terms on the right.
[3 marks]
Using the definition of the relationship between the interaction and Schroedinger pictures, this is

$$
\begin{aligned}
i\hbar\frac{\partial \hat{A}_I(t)}{\partial t} &= -\hat{H}_0\hat{A}_I(t) \\
&\quad + \exp\left(i\hat{H}_0 t/\hbar\right)\left(i\hbar\frac{\partial \hat{A}(t)}{\partial t}\right)\exp\left(-i\hat{H}_0 t/\hbar\right) \\
&\quad + \hat{A}_I(t)\hat{H}_0.
\end{aligned}
$$

The first and last terms combine into a commutator:

$$
i\hbar\frac{\partial \hat{A}_I(t)}{\partial t} = \left[\hat{A}_I(t), \hat{H}_0\right] + \exp\left(i\hat{H}_0 t/\hbar\right)\left(i\hbar\frac{\partial \hat{A}(t)}{\partial t}\right)\exp\left(-i\hat{H}_0 t/\hbar\right)
$$

[1 mark]
Note that the partial derivatives are also total derivatives; we could have used that throught.

$$
i\hbar\frac{\mathrm{d}\hat{A}_I(t)}{\mathrm{d}t} = \left[\hat{A}_I(t), \hat{H}_0\right] + i\hbar\exp\left(i\hat{H}_0 t/\hbar\right)\left(\frac{\mathrm{d}\hat{A}(t)}{\mathrm{d}t}\right)\exp\left(-i\hat{H}_0 t/\hbar\right).
$$

[1 mark]
Equivalently, we could again write

$$
i\hbar\frac{\mathrm{d}\hat{A}_I(t)}{\mathrm{d}t} = \left[\hat{A}_I(t), \hat{H}_0\right] + i\hbar\left(\frac{\mathrm{d}\hat{A}_S(t)}{\mathrm{d}t}\right)_I.
$$

3 Time evolution (assorted questions not in exam style)

NB you do not need to submit answers to these questions. Marks are provided simply to offer guidance on how long to spend on each question.
If you are happy with your answers to Q2, you might choose to work on Q3 in the problems class.

3.1 Equivalence of pictures (advanced)

We claimed before that the three pictures are equivalent, since

$$\langle\varphi_S(t)|\hat{A}_S|\psi_S(t)\rangle = \langle\varphi_H|\hat{A}_H(t)|\psi_H\rangle = \langle\varphi_I(t)|\hat{A}_I(t)|\psi_I(t)\rangle. \tag{33}$$

But we're now encountering objects of the form

$$\langle\varphi_I(t_2)|\hat{A}_I(t)|\psi_I(t_1)\rangle. \tag{34}$$

(i) Explain why the pictures must still be equivalent.
(ii) Consider the case where

$$\hat{H} = \hat{H}_0 \tag{35}$$

so that the interaction picture and Heisenberg pictures are equivalent. By defining

$$\langle n|\psi_S(t)\rangle = c_{S,n}(t) \tag{36}$$
$$\langle n|\psi_H\rangle = c_{H,n} \tag{37}$$

where

$$\hat{H}_0 = \sum_n E_n|n\rangle\langle n| \tag{38}$$

write

$$\langle\psi_S(t_2)|\psi_S(t_1)\rangle \tag{39}$$

in both the Schroedinger and Heisenberg pictures, and hence derive a relationship between the complex constants c.

[6 marks]

Schroedinger:

$$\begin{aligned}\langle\psi_S(t_2)|\psi_S(t_1)\rangle &= \sum_n \langle\psi_S(t_2)|n\rangle\langle n|\psi_S(t_1)\rangle \\ &= \sum_n c^*_{S,n}(t_2)\, c_{S,n}(t_1)\,.\end{aligned}$$

Heisenberg:

$$
\begin{aligned}
\langle \psi_S(t_2) | \psi_S(t_1) \rangle &= \langle \psi_S(0) | \exp\left(i\hat{H}_0 t_2/\hbar\right) \exp\left(-i\hat{H}_0 t_1/\hbar\right) |\psi_S(0)\rangle \\
&= \langle \psi_H | \exp\left(i\hat{H}_0 (t_2 - t_1)/\hbar\right) |\psi_H\rangle \\
&= \sum_{n,m} \langle \psi_H | m\rangle\langle m| \exp\left(i\hat{H}_0 (t_2 - t_1)/\hbar\right) |n\rangle\langle n|\psi_H\rangle \\
&= \sum_{n,m} c^*_{H,m} \langle m| \exp\left(i\hat{H}_0 (t_2 - t_1)/\hbar\right) |n\rangle c_{H,n} \\
&= \sum_{n,m} c^*_{H,m} \langle m|n\rangle \exp\left(iE_n (t_2 - t_1)/\hbar\right) c_{H,n} \\
&= \sum_{n,m} c^*_{H,m} \delta_{mn} \exp\left(iE_n (t_2 - t_1)/\hbar\right) c_{H,n} \\
&= \sum_{n} c^*_{H,n} c_{H,n} \exp\left(iE_n (t_2 - t_1)/\hbar\right)
\end{aligned}
$$

therefore

$$
c^*_{S,n}(t_2)\, c_{S,n}(t_1) = c^*_{H,n} c_{H,n} \exp\left(iE_n (t_2 - t_1)/\hbar\right).
$$

[Well, strictly we only require

$$
\sum_n \left(c^*_{S,n}(t_2)\, c_{S,n}(t_1) - c^*_{H,n} c_{H,n} \exp\left(iE_n (t_2 - t_1)/\hbar\right)\right) = 0]
$$

3.2 2nd order scattering

In the lecture notes, based on Ben Simons' notes, we considered the SHO (harmonic oscillator) potential

$$
\hat{H}_0 = \frac{1}{2} m\omega^2 \hat{x}^2 \tag{40}
$$

in the presence of the time-dependent potential

$$
\hat{V}(t) = -eE\hat{x} \exp\left(-t^2/\tau^2\right). \tag{41}
$$

Denoting the energy eigenstates

$$
\hat{H}_0 |n\rangle = \hbar\omega \left(n + \frac{1}{2}\right) |n\rangle \tag{42}
$$

We calculated the amplitude

$$
\langle 1(t_b) | 0(t_a)\rangle = \langle 1(t_b) | \hat{U}(t_a, t_b) | 0(t_b)\rangle \tag{43}
$$

where $t_a = -\infty$ and $t_b = +\infty$, to first order in the expansion of the time evolution operator. Now we will find

$$
\langle 2(t_b) | 0(t_a)\rangle.
$$

3.2.1 2nd order required

Show that to 1st order in the expansion of the time evolution operator

$$
\langle 2(t_b) | 0(t_a)\rangle = 0
$$

and therefore we need to go to second order.

[6 marks]

3.2.2 Setting up the integrals, part 1

Show that, to second order,

$$\langle 2_I(t_b)|0_I(t_a)\rangle = \frac{1}{\hbar^2}\int_{t_a}^{t_b} \mathrm{d}t' \int_{t_a}^{t'} \mathrm{d}t'' \langle 2_I(t_b)|\hat{V}_I(t')\,\hat{V}_I(t'')\,|0_I(t_b)\rangle. \tag{44}$$

[6 marks]

3.2.3 Setting up the integrals, part 2

Show that

$$\langle 2_I(t_b)|0_I(t_a)\rangle = \sqrt{2}\frac{e^2E^2}{2m\hbar\omega}\int_{t_a}^{t_b} \mathrm{d}t' \int_{t_a}^{t'} \mathrm{d}t'' \exp\left(-\frac{t'^2+t''^2}{\tau^2}\right)\exp\left(i\omega\left(t'+t''\right)\right). \tag{45}$$

[10 marks]

3.2.4 Doing the integrals, part 1

Draw the region of integration in the (t', t'') plane to explain why the change of variables

$$t_1 = \frac{t''+t'}{\sqrt{2}} \tag{46}$$

$$t_2 = \frac{t''-t'}{\sqrt{2}} \tag{47}$$

gives

$$\langle 2_I(\infty)|0_I(-\infty)\rangle = \sqrt{2}\frac{e^2E^2}{2m\hbar\omega}\int_{-\infty}^{0} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_1 \exp\left(-\frac{t_1^2+t_2^2}{\tau^2}+\sqrt{2}i\omega t_1\right). \tag{48}$$

[6 marks]

3.2.5 Doing the integrals, part 2

Evaluate the integrals to find the required probability.

[6 marks]