Lagrangian Classical Mechanics: Summer Recap

1. The (classical) simple harmonic oscillator in 1D describes a particle in a potential

$$V\left(x\right)=\frac{1}{2}m\omega^{2}x^{2}. \tag{1}$$

Explain why the action for a particle in a simple harmonic oscillator, between times $t=0$ and $t=T$, is given by

$$S_{\mathrm{SHO}}\left[x\right]=\frac{m}{2}\int_{0}^{T}\mathrm{d}t'\left(\dot{x}^{2}-\omega^{2}x^{2}\right). \tag{2}$$

[3 marks]

2. Derive the Euler Lagrange equations for a general action.

[5 marks]

3. Show that classical trajectories of the Harmonic oscillator $x_{c}\left(t\right)$ obey

$$\ddot{x}_{c}=-\omega^{2}x_{c}. \tag{3}$$

[3 marks]

4. Solve for the classical trajectory $x\left(t\right)$ assuming $x\left(t=0\right)=0$ and $x\left(t=T\right)=X$.

[3 marks]

5. Hence evaluate the action between times 0 and T, subject to these same boundary conditions. Check that your answer has the correct dimensions.

[3 marks]

6. Find the momentum p conjugate to the position x for the classical harmonic oscillator.

[2 marks]

7. By performing a Legendre transform on the Lagrangian $L\left(\dot{x},x\right)$, derive the Hamiltonian of the simple harmonic oscillator $H\left(p,x\right)$.

[3 marks]

8. Sketch the phase space trajectories for the classical harmonic oscillator.

[3 marks]