Lagrangian Classical Mechanics: Summer Recap

1. The (classical) simple harmonic oscillator in 1D describes a particle in a potential

$$V(x) = \frac{1}{2} m\omega^2 x^2. \tag{1}$$

Explain why the action for a particle in a simple harmonic oscillator, between times $t = 0$ and $t = T$, is given by

$$S_{\text{SHO}}[x] = \frac{m}{2} \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2 \right). \tag{2}$$

[3 marks]

$$S = \int \mathrm{d}t L$$

[1 mark]
and

$$L = T - V$$

[1 mark]
so

$$\begin{aligned} S &= \int_0^T \mathrm{d}t \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \right) \\ &= \frac{m}{2} \int_0^T \mathrm{d}t \left(\dot{x}^2 - \omega^2 x^2 \right) \end{aligned}$$

as required.
[1 mark]

2. Derive the Euler Lagrange equations for a general action.

[5 marks]

We must extremise the action:

$$\left(\frac{\partial S[q_i + \lambda \epsilon_i]}{\partial \lambda} \right)_{q_i, \epsilon_i} \Bigg|_{\lambda=0} = 0. \tag{3}$$

The vector function $\epsilon_i(t)$ parameterises a variation away from the classical path $q_i(t)$. We require that the variation is zero at the start and end points of the trajectory:

$$\epsilon_i(t_0) = \epsilon_i(t_f) = 0. \tag{4}$$

Specifically:

$$S\left[q_i+\lambda\epsilon_i\right] = \int_{t_0}^{t_f} L\left(q_i+\lambda\epsilon_i, \dot{q}_i+\lambda\dot{\epsilon}_i, t\right)\mathrm{d}t \tag{5}$$

$\Downarrow$ chain rule

$$\left(\frac{\partial S\left[q_i+\lambda\epsilon_i\right]}{\partial\lambda}\right)_{q_i,\epsilon_i} = \int_{t_0}^{t_f}\left\{\left(\frac{\partial L}{\partial q_i}\right)_{\dot{q}_i,t}\left(\frac{\partial\left(q_i+\lambda\epsilon_i\right)}{\partial\lambda}\right)_{q_i,\epsilon_i}\right. \tag{6}$$

$$\left.+\left(\frac{\partial L}{\partial\dot{q}_i}\right)_{q_i,t}\left(\frac{\partial\left(\dot{q}_i+\lambda\dot{\epsilon}_i\right)}{\partial\lambda}\right)_{q_i,\epsilon_i}+\cancel{\left(\frac{\partial L}{\partial t}\right)_{q_i,\dot{q}_i}\left(\frac{\partial t}{\partial\lambda}\right)_{q_i,\epsilon_i}}\right\}\mathrm{d}t \tag{7}$$

$$= \int_{t_0}^{t_f}\left\{\left(\frac{\partial L}{\partial q_i}\right)_{\dot{q}_i,t}\epsilon_i+\left(\frac{\partial L}{\partial\dot{q}_i}\right)_{q_i,t}\dot{\epsilon}_i\right\}\mathrm{d}t \tag{8}$$

Now integrate the $\dot{\epsilon}$ term by parts in Eq 8:

$$\left(\frac{\partial S\left[q_i+\lambda\epsilon_i\right]}{\partial\lambda}\right)_{q_i,\epsilon_i} = \int_{t_0}^{t_f}\left\{\left(\frac{\partial L}{\partial q_i}\right)_{\dot{q}_i,t}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)_{q_i,t}\right\}\epsilon_i\mathrm{d}t+\left[\left(\frac{\partial L}{\partial\dot{q}_i}\right)_{q_i,t}\epsilon_i\right]_{t_0}^{t_f} \tag{9}$$

but the boundary term vanishes by assumption. Applying the principle of least action, Eq 3, we require

$$\left.\left(\frac{\partial S\left[q_i+\lambda\epsilon_i\right]}{\partial\lambda}\right)_{q_i,\epsilon_i}\right|_{\lambda=0} = 0 = \int_{t_0}^{t_f}\left\{\left(\frac{\partial L}{\partial q_i}\right)_{\dot{q}_i,t}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)_{q_i,t}\right\}\epsilon_i\mathrm{d}t. \tag{10}$$

This is true for all $\epsilon_i(t)$ (since this arbitrary function has not been specified). This gives

$$\left(\frac{\partial L}{\partial q_i}\right)_{\dot{q}_i,t}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)_{q_i,t}=0. \tag{11}$$

3. Show that classical trajectories of the Harmonic oscillator $x_c(t)$ obey

$$\ddot{x}_c = -\omega^2 x_c. \tag{12}$$

[3 marks]

Classical trajectories obey the Euler Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

[1 mark]
giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right)-\left(-m\omega^2 x\right)=0$$

[2 marks], one for each part, or

$$m\ddot{x}+m\omega^2 x=0$$

which simplifies to the stated expression.

4. Solve for the classical trajectory $x(t)$ assuming $x(t=0)=0$ and $x(t=T)=X$.

[3 marks]

The general solution is

$$x_c = A\sin(\omega t) + B\cos(\omega t)$$

[1 mark]
and substituting the boundary conditions gives

$$x = X\frac{\sin(\omega t)}{\sin(\omega T)}$$

[2 marks].

5. Hence evaluate the action between times 0 and T, subject to these same boundary conditions. Check that your answer has the correct dimensions.

[3 marks]

Insert the expression into the action:

$$\begin{aligned}
S_{\text{SHO}}[x] &= \frac{m}{2}\int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right) \\
&= \frac{mX^2\omega^2}{2\sin^2(\omega T)}\int_0^T \mathrm{d}t' \left(\cos^2(\omega t) - \sin^2(\omega t)\right) \\
&= \frac{mX^2\omega^2}{2\sin^2(\omega T)}\int_0^T \mathrm{d}t' \left(\cos(2\omega t)\right) \\
&= \frac{mX^2\omega}{4\sin^2(\omega T)}\left[\sin(2\omega t)\right]_0^T \\
&= \frac{mX^2\omega\sin(2\omega T)}{4\sin^2(\omega T)} \\
&= \frac{1}{2}m\omega X^2\cot(\omega T)\,.
\end{aligned}$$

[2 marks]
To check the dimensions, note that

$$[S] = \mathbb{E}\mathbb{T}$$

and since

$$\left[\frac{1}{2}m\omega X^2\right] = \left[\frac{1}{2}m\omega^2 X^2\right]\left[\omega^{-1}\right] = \mathbb{E}\mathbb{T}$$

this works out. Strictly there are some dimensionless radians in there, but they actually make sense if thought about systematically (e.g. $E = hf = \hbar\omega$, and h and $\hbar$ have the same units).
[1 mark]

6. Find the momentum p conjugate to the position x for the classical harmonic oscillator.

[2 marks]

$$p \triangleq \frac{\partial L}{\partial \dot{x}}$$

[1 mark]
giving

$$p = m\dot{x}$$

[1 mark]

7. By performing a Legendre transform on the Lagrangian $L(\dot{x}, x)$, derive the Hamiltonian of the simple harmonic oscillator $H(p, x)$.

[3 marks]

To calculate the Hamiltonian from the Lagrangian:

$$H(x, p) = p\dot{x} - L(x, \dot{x})$$

[1 mark]
giving

$$H(x, p) = p\dot{x} - \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right)$$

[1 mark]
and we now need to eliminate $\dot{x}$ in favour of p:

$$\begin{aligned} H(x, p) &= p^2/m - \left(\frac{1}{2}p^2/m - \frac{1}{2}m\omega^2 x^2\right) \\ &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \end{aligned}$$

[1 mark]

8. Sketch the phase space trajectories for the classical harmonic oscillator.

[3 marks]

The trajectories are clockwise **[1 mark]** circles **[1 mark]** centred on the origin **[1 mark]**.