Quantum Fields and Particles: Problem Sets

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Problems sets 1, 2, and 3 are due on Thursday 10am on weeks 2, 3, and 4 respectively.

We will go through the solutions in the problems classes on Thursdays 10-11am in weeks 3, 4, and 5 respectively.

The marking scheme for coursework will be unusual! You will receive full marks on a problem set if you attempt every question, *regardless of whether you get it right*.

An attempt is defined as either:
(i) a logical sequence of reasoning leading to a proposed answer, or
(ii) a list of two specific locations in textbooks (paragraphs/subsections) you believe to be the most relevant to answering the question, and a written explanation of why you haven't understood from them how to achieve (i).

If you opt for (ii) at any point, please provide a cover sheet for your problem set highlighting where you do so.

Marks are indicated for each question only as a guide to the amount of time you should spend on each part. Each problem set is pass (100%, all part-questions attempted) / fail (0%, one or more part-questions not attempted).

Marking: you will be assigned to groups of 3. One of you will receive detailed feedback on each problem set each week. During the problems class, it will be that person's job to ensure that all three understand the solutions to every problem.

1 Canonical Quantization of Fields

1.1

Consider a 1D chain of atoms, with two different atoms per unit cell. The total unit cell length is a. The mass of the atoms with displacements $u_n(t)$ is m, and the mass of atoms with displacements $v_n(t)$ is $M > m$. Explain with the aid of a diagram why the classical motion is described by

$$m\ddot{u}_n = -K(u_n - v_n) + K(v_{n+1} - u_n) \tag{1}$$
$$M\ddot{v}_n = K(u_n - v_n) - K(v_n - u_{n-1}). \tag{2}$$

[3 marks]

1.2

1.2.1

Show that the normal modes of this chain are given by

$$\omega_\pm^2 = \frac{K}{\mu}\left(1 \pm \sqrt{1 - \frac{4\mu^2}{mM}\left(\sin^2(ka/2)\right)}\right) \tag{3}$$

where we have defined the reduced mass

$$\mu \triangleq \frac{mM}{m+M}. \tag{4}$$

[5 marks]

1.2.2

The two branches of the dispersion are called the acoustic branch $\omega_-(k)$ and the optical branch $\omega_+(k)$. Find their dispersions to leading order in k around $k = 0$.

[4 marks]

1.2.3

Hence find the speed of sound c of the acoustic branch:

$$c \triangleq \lim_{k\to 0} \frac{\omega_-}{k} \tag{5}$$

[2 marks]

and the *mass gap* Δ of the optical branch:

$$\Delta \triangleq \omega_+(k = 0). \tag{6}$$

[2 marks]

Show that to leading order in k,

$$\omega_- = ck$$
$$\omega_+ = \sqrt{\Delta^2 - c^2k^2}.$$

[2 marks]

1.2.4

Find the gap $\omega_+ - \omega_-$ at $k = \pi/a$. Show that it closes when $m = M$, and explain this result.

[2 marks]

1.2.5

Sketch $\omega_\pm$ over $k \in [-\pi/a, \pi/a]$ marking on values at key points.

1.3

Show that the Lagrangian

$$L = \sum_{n=1}^{N} \frac{1}{2} m \dot{u}_n^2 + \frac{1}{2} M \dot{v}_n^2 - \frac{K}{2} \left\{ (u_n - v_n)^2 + (v_{n+1} - u_n)^2 \right\} \tag{7}$$

leads to the correct equations of motion.

[4 marks]

Hint: it is easiest tp derive separate equations of motion for $\dot{u}$,u holding $\dot{v}$,v constant and vice versa.

1.4

1.4.1

Take the continuum limit to find classical fields using the substitutions

$$u_n(t) \to U(x,t) \tag{8}$$

$$v_n(t) \to V(x,t) \tag{9}$$

where $x = na$ at lattice sites. Using a Taylor expansion

$$v_{n+1} \to V(x+\epsilon) = V(x) + \epsilon V'(x) + \frac{1}{2}\epsilon^2 V''(x) + \mathcal{O}(\epsilon^3)$$

for v_{n+1}, show that the resulting coarse-grained Lagrangian is

$$L = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a} \left[\frac{1}{2} m \dot{U}^2 + \frac{1}{2} M \dot{V}^2 - \frac{K}{2} \left\{ 2(U-V)^2 + (\epsilon V')^2 + \left(\frac{1}{2}\epsilon^2 V''\right)^2 + 2\epsilon (V-U) V' + \epsilon^2 V'' (V-U) + \epsilon^3 V' V'' \right\} \right]. \tag{10}$$

[4 marks]

1.4.2

Explain why this is equal to

$$L = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a} \left[\frac{1}{2} m \dot{U}^2 + \frac{1}{2} M \dot{V}^2 - \frac{K}{2} \left\{ 2(U-V)^2 + \left(\frac{1}{2}\epsilon^2 V''\right)^2 - U\left(2\epsilon V' + \epsilon^2 V''\right) \right\} \right]. \tag{11}$$

[2 marks]

1.5

We would like to decouple the fields U and V. To do so we can use transformed fields

$$U = \alpha\Phi + \beta\Psi \tag{12}$$
$$V = \gamma\Phi + \delta\Psi. \tag{13}$$

1.5.1

Find a condition on α, β, γ, δ required for the kinetic terms to remain decoupled.

[2 marks]

1.5.2

One possible choice is

$$\alpha = \gamma = 1 \tag{14}$$
$$\beta = \mu/m \tag{15}$$
$$\delta = -\mu/M \tag{16}$$

giving

$$U = \Phi + \frac{\mu}{m}\Psi \tag{17}$$
$$V = \Phi - \frac{\mu}{M}\Psi. \tag{18}$$

Show that Ψ and Φ are simply the relative and centre-of-mass co-ordinates.

[2 marks]

1.5.3

Using these new fields, and dropping terms of order $> \epsilon^2$, show that the Lagrangian density can be written

$$\mathcal{L} = \mathcal{L}_{\text{acoustic}} + \mathcal{L}_{\text{optical}} + \mathcal{L}_{\text{int}} \tag{19}$$

where

$$\mathcal{L}_{\text{acoustic}} = \frac{m+M}{2}\dot{\Phi}^2 - \frac{K}{2}\left(\epsilon\Phi'\right)^2 \tag{20}$$
$$\mathcal{L}_{\text{optical}} = \frac{\mu}{2}\dot{\Psi}^2 + \frac{K}{2}\left(\frac{\mu}{m+M}\right)\left(\epsilon\Psi'\right)^2 - K\Psi^2 \tag{21}$$

and $\mathcal{L}_{\text{int}}$ is an interaction term you should state (which we will subsequently neglect).

[2 marks]

1.6

Consider the general action for a relativistic massive real scalar field, with $\hbar$ and c written explicitly (assuming the field itself is dimensionless):

$$S\left[\varphi\right] = \frac{\hbar}{c}\int \mathrm{d}t \int \mathrm{d}x \left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\left(c\nabla\varphi\right)^2 - \frac{1}{2}\frac{m^2c^4}{\hbar^2}\varphi^2\right). \tag{22}$$

1.6.1

By varying the action with respect to the field, show that the Euler Lagrange equation is the Klein Gordon equation

$$\ddot{\varphi} - c^2\nabla^2\varphi + \frac{m^2c^4}{\hbar^2}\varphi = 0. \tag{23}$$

[6 marks]

1.6.2

Show that for a massless field, the Klein Gordon equation reduces to the wave equation for a wave of speed c.

[1 mark]

1.6.3

Hence, show that $\mathcal{L}_{\text{acoustic}}$ corresponds to a massless scalar field, and find the speed of sound.

[2 marks]

1.6.4

Find the length scaling ϵ such that the speed of sound in the continuum model matches that in the microscopic model.

[2 marks]

1.6.5

Show that $\mathcal{L}_{\text{optical}}$ corresponds to a massive scalar field. Show that the mass, in natural units $\hbar = c = 1$, is equal to the mass gap of the optical branch of the dispersion.

[2 marks]

1.7

1.7.1

Find Π, the momentum canonically conjugate to Ψ.

[1 mark]

1.7.2

Find the classical field Hamiltonian corresponding to $\mathcal{L}_{\text{optical}}$.

[4 marks]

1.8

We saw in the lectures how to canonically quantize the acoustic branch. We will now quantize the optical branch.

1.8.1

Write the canonically quantized field Hamiltonian.

[1 mark]

1.8.2

Using the Fourier transformed field operators (we will use the same symbols for the fields and their Fourier transforms, to save on notation):

$$\hat{\Psi}_{x,t} = \int \frac{\mathrm{d}k}{2\pi} \exp(ikx)\, \hat{\Psi}_{k,t}$$
$$\hat{\Pi}_{x,t} = \int \frac{\mathrm{d}k}{2\pi} \exp(-ikx)\, \hat{\Pi}_{k,t}$$

show that the Hamiltonian can be written

$$\hat{H} = \int \frac{\mathrm{d}k}{2\pi} \left\{ \frac{1}{2\mu} \hat{\Pi}_k \hat{\Pi}_{-k} + \frac{1}{2} \mu \omega_k^2 \hat{\Psi}_k \hat{\Psi}_{-k} \right\}$$

for an ω_k you should specify.

[4 marks]

1.8.3

Show that the lengthscale ϵ you found in Question 1.6.4 causes the optical dispersion in the continuum model to match that which you found in the microscopic model in Question 1.2.2, at small k.

[2 marks]

1.8.4

Sketch the dispersions of the acoustic and optical fields over the original dispersions.

[2 marks]

1.9

Find the second-quantized normal-ordered Hamiltonian $:\hat{H}:$ for the optical phonon field by rewriting in terms of creation and annihilation operators

$$\hat{\Psi}_k = \frac{1}{\sqrt{2\mu\omega_k}} \left(\hat{a}_k + \hat{a}^\dagger_{-k} \right) \tag{24}$$
$$\hat{\Pi}_k = i\sqrt{\frac{\mu\omega_k}{2}} \left(\hat{a}_k - \hat{a}^\dagger_{-k} \right). \tag{25}$$

[4 marks]

2 Path Integral Field Quantization

In this problem set we will derive the propagator of the photon.

2.1

- Explain the philosophy behind path integral quantum mechanics, including how non-commutation arises. **[3 marks]**
- Why is a Lagrangian formulation of quantum mechanics especially advantageous for QFT? **[2 marks]**
- Quantum Mechanics is sometimes said to be a 0+1D quantum field theory. Explain why. **[3 marks]**

2.2

2.2.1

Using the chain rule, derive the Euler Lagrange equations for a general Lagrange density that is a function of a Lorentz scalar field: $\mathcal{L}(\varphi, \partial_\mu \varphi)$.

[5 marks]

2.2.2

Derive the Euler Lagrange equations for a Lagrange density that is a function of a Lorentz vector field: $\mathcal{L}(A^\nu, \partial_\mu A^\nu)$.

[5 marks]

2.3

The photon is governed by Maxwell theory, whose Lagrange density is

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{26}$$

where

$$F^{\mu\nu} \triangleq \partial^\mu A^\nu - \partial^\nu A^\mu . \tag{27}$$

2.3.1

Find the Euler Lagrange equations of Maxwell theory.
Hint: you will need to use the fact that

$$\frac{\partial (\partial_\nu A^\mu)}{\partial (\partial_\alpha A^\beta)} = \delta^\alpha_\nu \delta^\mu_\beta \tag{28}$$

where δ^α_ν is a Kronecker delta. Additionally, you will need to make use of the Minkowski metric $\eta^{\mu\nu}$ to raise and lower indices as appropriate.

[6 marks]

2.3.2

Show that $F^{\mu\nu}$ is invariant to *local gauge transformations*

$$A^{\mu} \to A^{\mu} + \partial^{\mu} \lambda(x) \tag{29}$$

for arbitrary $\lambda(x)$.

[2 marks]

2.4

Following the lectures, we can attempt to find the momentum-space photon propagator $\widetilde{G}_{\mu\nu}$ by rewriting the Fourier transformed action in the form

$$S_{\text{Maxwell}} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{2} \widetilde{A}^{\mu} \widetilde{G}^{-1}_{\mu\nu} \widetilde{A}^{\nu}. \tag{30}$$

Show that this gives

$$\widetilde{G}^{-1}_{\mu\nu} = p_{\mu} p_{\nu} - \eta_{\mu\nu} p^2. \tag{31}$$

[6 marks]

2.5

However, $\widetilde{G}^{-1}_{\mu\nu}$ cannot be inverted. To show this, find an eigenstate of $\widetilde{G}^{-1}_{\mu\nu}$ with zero eigenvalue, and explain the relevance.

[4 marks]

2.6

To see where the issue arises, state the number of degrees of freedom of the real vector field $\widetilde{A}^{\mu}(\omega, \boldsymbol{k})$, and the number of degrees of freedom of a photon with wavevector $\boldsymbol{k}$ and energy ω.

[2 marks]

2.7

To proceed, we must restrict the freedom of A^{μ} by making a *gauge choice*. Consider the modified action

$$S^{\xi}_{\text{Maxwell}} = \int \mathrm{d}^4 x \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi} \left(\partial^{\mu} A_{\mu}\right)^2 \right\}. \tag{32}$$

The new term can be thought of as a Lagrange multiplier: taking the limit $\xi \to 0$ enforces the gauge choice

$$\partial^{\mu} A_{\mu} = 0 \tag{33}$$

which is the *Landau gauge*. We do the calculation with arbitrary ξ, then take the limit $\xi \to 0$ at the end. Find $\widetilde{G}^{-1}_{\mu\nu}$ for this modified action.

[4 marks]

2.8

To invert and find the momentum-space Green's function, identify $\widetilde{G}^{\mu\nu}$ such that

$$\widetilde{G}^{\mu\lambda}\widetilde{G}^{-1}_{\lambda\nu} = \delta^{\mu}_{\nu}. \tag{34}$$

Hint: make an educated guess at an ansatz, keeping the dimensions in mind. Recall that

$$\eta^{\mu\lambda}\eta_{\lambda\nu} = \eta^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}. \tag{35}$$

[8 marks]

2.9

We motivated the ξ term by saying that it enforces the Landau gauge when $\xi \to 0$. Other gauge choices can be made with other values of ξ (the term added to the action turns out not to affect physical results). Another common choice is the Feynman gauge, $\xi = 1$. Show that the real-space photon propagator in the Feynman gauge is

$$G^{\mu\lambda} = -\eta^{\mu\lambda}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\exp{(ip\cdot x)}}{p^2 + i\epsilon} \tag{36}$$

where ϵ is a term you should explain.

[4 marks]

3 Interacting Quantum Fields

In the lectures we have focussed on φ^4 theory. A simpler theory is φ^3; it is, unfortunately, physically unrealisable. Still, we can look at some of its properties.

3.1

Consider the Lagrange density for a real scalar field:

$$\mathcal{L} = \mathcal{L}_{\text{Klein Gordon}} + \mathcal{L}_{\text{int}} \tag{37}$$

with

$$\mathcal{L}_{\text{int}} = \frac{g}{\mathcal{N}}\varphi^3 \tag{38}$$

where $\mathcal{N}$ is an integer which we will determine in this question.

3.1.1

Draw the full set of Feynman diagrams (including disconnected diagrams) that contribute to the 3-point function $\langle\varphi_1\varphi_2\varphi_3\rangle$ at linear order in g.

[4 marks]

3.1.2

Explain what is meant by a 1PI diagram, and why the concept is useful.

[3 marks]

3.1.3

Using Wick's theorem, derive the 3-point function $\langle\varphi_1\varphi_2\varphi_3\rangle$ explicitly to order g in terms of non-interacting Klein Gordon propagators. Hence suggest a suitable normalisation $\mathcal{N}$.

[8 marks]

3.1.4

Assuming your choice of $\mathcal{N}$, write down the real-space Feynman rules for φ^3 theory by specifying the algebraic equivalents to:

- an internal vertex at x
- a line connecting vertices at x and y.

[4 marks]

3.1.5

Write down the full $\langle\varphi_1\varphi_2\varphi_3\rangle$ (including disconnected diagrams) to order g in terms of Green's functions directly from your Feynman diagrams. Explain the symmetry factors of each.

[8 marks]

3.2

In φ^3 theory, only even powers of g contribute to N-point functions for even N, and only odd powers of g contribute for odd N. Explain this:

3.2.1

Algebraically

[2 marks]

3.2.2

Diagrammatically (i.e. using Feynman diagrams).

[2 marks]

3.3

Sketch all the 1PI vacuum diagrams of φ^3 theory to order g^4.

[3 marks]

3.4

3.4.1

Identify the 1PI diagram contributing to the renormalisation of the propagator up to order g^2.

[2 marks]

3.4.2

Calling the amputated 1PI diagram Π, write an algebraic Dyson series for the renormalisation of the momentum space propagator $\widetilde{G}$ to infinite order.

[4 marks]

3.4.3

Rewrite the Dyson series calculation entirely diagrammatically. You can denote $\widetilde{G}_{\text{ren}}$ however you like (e.g. a double line).

[2 marks]

3.4.4

Use the momentum space propagator

$$\widetilde{G} = \frac{i}{p^2 - m^2 + i\epsilon}$$

to evaluate the Dyson series, showing that the renormalised propagator is

$$\widetilde{G}_{\text{ren}} = \frac{i}{p^2 - m^2 - i\Pi}. \tag{39}$$

[4 marks]