Fermi Problems

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The following problems are to accompany the first year class on Estimation and Dimensional Analysis. There are two excellent references freely available online (linked on my website):

- "Order-of-Magnitude Physics: Understanding the World with Dimensional Analysis, Guesswork, and White Lies" by P. Goldreich, S. Mahajan, and S. Phinney
- "Modern Physics from an Elementary Point of View" by V. F. Weisskopf.

1 Estimation

1.1 Estimate the following quantities

First, guess. Second, estimate by breaking down quantities you need to know into those you can make a confident guess about. For example:

$$\#\,piano\,tuners\,in\,Chicago = \frac{\#\,piano\,tuners}{person} \cdot \frac{\#\,people}{building} \cdot \#\,buildings\,in\,Chicago$$

etc. etc. (read $\frac{a}{b}$ as 'a per b'). When your group has an answer write out your chain of reasoning as I've done here.

1.1.1 Classic Problems

- The number of piano tuners in Chicago
- The number of aeroplanes in the sky
- The maximum height of a mountain
- The strength of gravity on Krypton, back when Superman could only jump over buildings
- The amount of money in an ATM
- How many times can you fold a piece of paper?
- In the classic film, Bond and Goldfinger engage in a bit of Fermi estimation. Put reasonable numbers back into the dialogue concerning breaking into the gold reserve at Fort Knox:

  [Goldfinger] You are unusually well informed, Mr Bond.

  [Bond] You'll kill (α) people uselessly.

  [Goldfinger] Ha! American motorists kill that many every (β) years.

[Bond] Yes, well... I've worked out a few statistics of my own. \$(γ) billion in gold bullion weighs (δ) tons. (ϵ) men would take (ϕ) days to load it onto (η) trucks. At the most, you'll have (κ) hours before the army, navy, air force, marines move in and make you put it back.

[Goldfinger] Who mentioned anything about removing it?

1.1.2 Physics

- The radius of a neutron star
- If we switch over to purely relying on tidal power, how long before the moon leaves its orbit?[1]
- It is often stated that water always drains anticlockwise in the northern hemisphere. Is this true? Estimate the size of the Coriolis force compared to gravitational fluctuations. How about the flow induced by choice of tap?
- The size of an atom
- In ARPES experiments electrons are deflected through a few degrees as part of the measurement. Is it important to know the Earth's magnetic field direction at the detector? How far away should the neighbouring high magnetic field lab be built?
- Will X-ray diffraction be a good probe of crystal structure right up to the material's melting point?
- What RAM would a computer need to simulate the universe? How about a quantum computer?
- How many glow in the dark watches would be needed to pose a threat of radiation poisoning? How many to power a city in place of a nuclear power plant?

1.1.3 My Recent Experiences

- When submitting jobs to the supercomputer you have to state how long the job will take. If you guess too low the code will terminate with no output. Significantly too high and you'll be de-prioritised in the queue. How do you estimate it if the code contains 4 nested integrals? How many nested integrals can plausibly be coded at all?
- I looked up the owner of the aforementioned ARPES machine before visiting his lab. He has over 130 papers. Is this impressive?
- My friend's Dad is planning to walk the longest straight line possible in Britain. He emailed to ask what the error is in using an OS map's straight line rather than taking a true 'great circle'. Should he invest in an expensive globe?
- My neighbour asked to run the power cables for a rig at St Paul's carnival into my living room. How much would it cost?
- We have damp in our house, and our cheapskate landlord wants to have us run a dehumidifier rather than fix the problem. How much should we charge him to do so?

[1] Thanks to Prof. J. H. Hannay for this example.

1.1.4 Paranormal

- The plausibility of telepathy via electromagnetic waves
- There is a phenomenon known as Streetlight Interference (SLI) in which people believe their presence causes a particular streetlight to turn off or on. By considering a model of a faulty streetlight estimate whether the effect is likely to be real.
- Homeopathic concentrations are measured on a C-scale, with n-C being 100^{-n} weaker than the original. A recommended potency (higher 'potency' = lower concentration) is 30C. Is homeopathy pseudoscience?
- Is the Loch Ness monster in fact a family of Plesioaurs?
- A friend proposes that the soul may be an electromagnetic phenomenon surviving our body's death, and that the effect may be measurable. I claim the signal-to-noise ratio would be 'absurdly small', but I haven't checked. Check.
- Could aliens have visited Earth?

1.2 One piece of information game

Repeat the previous exercise, but this time (after submitting your estimate) you will be allowed one piece of information retrieved from the internet to refine your guess. What will it be, and does your estimate change significantly?

1.3 'Albert Hall' estimates

When presenting scientific data to non-scientists an analogy is often drawn, frequently involving the Royal Albert Hall. To quote that great source of all scientific knowledge, The Viz:

> "Facts have their own units of measurement. Tall things are not measured in feet and inches, or even metres, but in double decker buses, Nelson's columns, and Eiffel Towers. Weights are measured in elephants, jumps by insects are measured in men jumping over St Paul's Cathedrals, and loud noises are measured in Concordes taking off. Multiple things that could be laid end-to-end are measured in terms of how many times they would circle the globe or reach the moon and back."

Make Albert Hall estimates for:

- The relative size of the atom and its nucleus
- Your deBroglie wavelength
- The number of atoms in a cucumber
- The number of stars in a galaxy
- The relative size of the Earth and the Sun
- The relative strength of gravity and electromagnetism (on a lengthscale of your choice)
- The distance to Alpha Centauri
- [Think of your own examples]

1.4 Units

- Look back at your reasoning chains. Check the units cancel correctly.
- Quantum effects are important in a system if $\hbar$ is a reasonable scale of the action. Are they likely to be important for: a tennis ball? Swimming bacteria? Snowflake growth? Capacitors? Scanning tunneling microscopes? [think of your own counterintuitive examples]
- Relativistic effects are important if c is a reasonable scale of velocity. Are they likely to be important for: a rocket? A person in freefall? A person in freefall towards a black hole? An electron contributing to current in a metal? An electron in an atom? A nucleon in an atom? [think of your own counterintuitive examples]

2 Dimensional Analysis

2.1 Axioms

I think dimensional analysis is covered by these three axioms (dimensions of x are denoted $[x]$):

1. $\forall x+y,\ [x]=[y]$
2. $\forall f(x),\ [x]=0$
3. $[xy]=[x][y]$

Do you agree? Can you deduce (2) from (1)?

2.2 State the SI units

2.3 $\mathbb{L},\mathbb{T},\mathbb{M}$ decomposition

2.3.1 Simple Equations

As a warmup list some simple equations you may use to convert dimensions (*e.g.* $E=mc^2$, $F=ma$).

2.3.2 Decomposition

- Decompose the following quantities into dimensions of length $\mathbb{L}$, time $\mathbb{T}$, and mass $\mathbb{M}$: force, energy, momentum, $\hbar$, c, pressure, dB, temperature, entropy. Take $[k_B]$ to be dimensionless (a subtle point worth discussing).
- Pick a physical quantity which has each dimension and estimate its value in SI units.
- Electromagnetic quantities are a little tricky. For the interested reader there is an excellent review in the appendix of J. D. Jackson's *Classical Electrodynamics*[2]. One way to deal with the problem is to introduce an additional dimension for electric charge, call it $\mathbb{Q}$. Decompose electric current, electric field strength, magnetic field strength, and capacitance into $\mathbb{L}$, $\mathbb{T}$, $\mathbb{M}$, $\mathbb{Q}$.

[2]Thanks to S. Montes Valencia for making this point clear and for providing the reference. There is a further subtlety here, in that electric charge is fundamentally quantized. We don't know why this is.

2.3.3 Decomposition game

As a group think of a difficult unit or quantity to convert and pass it as a challenge to the next group. If they succeed they get 5 points. If they fail you get 5 points. If they fail they can call you out and challenge you to decompose it. If you succeed you get 10 points, if you fail they get 10 points.

2.3.4 Dimensional analysis

- Put the (multiple) $\hbar$ and c back into the right hand side of Fermi's Golden Rule:

$$\frac{1}{\tau} \propto G_F^2 \left|\mathcal{M}\right|^2 Q_0^5$$

where $[\tau] = time$, $[G_F] = Energy \cdot Volume$, $[Q_0] = Energy$, and $[\mathcal{M}] = 0$ (dimensionless).

- Put the multiple $\hbar$ and c back into the Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$$

where $[\gamma] = 0$ (dimensionless) and $[\partial] = \mathbb{L}^{-1}$.

- Put a consistent set of c back into Maxwell's equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho & \nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{B} &= \mathbf{j} + \dot{\mathbf{E}} \end{aligned}$$

where $[\nabla] = \mathbb{L}^{-1}$, ρ is charge density, and $\mathbf{j}$ is current density.

2.4 (Advanced) Decomposition into $\mathbb{E}$

We could restrict further to using only one dimension. Choosing energy $\mathbb{E}$, we can say for example $[E] = [m] = 1$, $[x] = [t] = -1$, where the number indicates the power of $\mathbb{E}$ (no ambiguity since we only have one dimension). Calculate the dimensions of the following fields given that $[S] = 0$ and $[\partial] = 1$.

- $S = \int \mathrm{d}^4x \left\{\frac{1}{2}\left(\partial\varphi\right)^2\right\}$
- $S = \int \mathrm{d}^4x \left\{\frac{1}{2}\left(\partial\varphi\right)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{4!}\varphi^4\right\}$ (g and m are not fields)
- $S = \int \mathrm{d}^3x \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho$ ($[\epsilon]$=0)
- $S = \int \mathrm{d}^3x \epsilon_{\mu\nu\rho}\left(A^\mu \partial^\nu A^\rho + gA^\mu A^\nu A^\rho\right)$
- $S = \int \mathrm{d}^4x \psi^\dagger \left(i\partial_t - H\right)\psi$
- $S = \int \mathrm{d}^3x \psi^\dagger \left(i\partial_t - H\right)\psi$
- $S = \int \mathrm{d}^4x \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$
- $S = \int \mathrm{d}^4x \left\{-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\mu - \partial^\nu A^\mu\right) + \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi - gA_\mu \overline{\psi}\gamma^\mu \psi\right\}$

If the coupling constant g has dimension $[g] > 0$ the theory is said to be 'relevant'. If $[g] < 0$ it is irrelevant, and if $[g] = 0$ it is marginal. Classify the three cases above.