Fourier Series

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As I don't know of a particularly good reference for this topic, and because I made a mistake in the class today which I imagine was fairly confusing, I've written the following notes to clarify how we do Fourier series, from a physicist's point of view. None of this stuff will be in the exam, for which you are free to quote results as you like - but if you put the time in to really understanding the basics at this stage you will not regret it in later years (when the exams count towards your degree!).

Say we'd like to decompose a function $f(x)$ as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $x \in [-L, L)$. Outside that domain the function repeats. The standard result for calculating the coefficients is

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \end{aligned}$$

I'll show where this comes from, and will explain how this whole process is really no more tricky than decomposing a vector into basis vectors.

The trick is that the functions $\left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}$ are 'orthogonal' over the domain $x \in [-L, L)$. This means that if you multiply two different ones together and integrate from $-L$ to L the result will be zero. I'll prove a slightly stronger result here, though, that $\left\{\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right), \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right)\right\}$ form an 'orthonormal basis' (orthogonal and normalized) over this domain.

First, some vector notation. You've probably seen vectors written as $\mathbf{u}$, $\underline{u}$, $\vec{u}$. There's another alternative (invented by Bristol's most famous alumnus Paul Dirac) where vectors are written $|u\rangle$. We define the transpose of the vector, $\mathbf{u}^T = \langle u|$. The nice thing about this notation is that inner products (the dot product for usual vectors) can be written as $\mathbf{u} \cdot \mathbf{v} = \langle u|v\rangle$. When vectors become more complicated, by being composed of complex numbers or having an infinite number of components for example, Dirac's notation treats them all exactly the same, $\langle u|v\rangle$.

An additonal thing we'll need is an 'inner product', which is the more general form of a dot product for vectors. For more information you can look up 'inner product spaces' on wikipedia or in a textbook - these are defined by a small set of rules, and if something obeys these rules it's an inner product. The normal dot product of two vectors obeys the rules - you can think of an inner product as a generalization. For normal vectors, then, the inner (or dot) product is given by

$$\langle u|v\rangle = \sum_n u_n v_n$$

(where u_n is here the n^{th} element of $|u\rangle$) or, if the vectors are complex,

$$\langle u|v\rangle = \sum_n u_n^* v_n.$$

Note that for complex numbers, $\langle v|u\rangle = (\langle u|v\rangle)^*$. As you'll recall of the dot product, the inner product gives the projection of one vector along another. We say in the above cases that we have found the projection of $|v\rangle$ in the $\langle u|$ basis.

Now, to keep everything completely correct, rather than over-simplified, the next bit will be a little conceptually difficult. Mathematically it's not too hard, but it requires you to accept some abstractions. We can write a function, say $\frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right)$, as the projection of a vector $|c_n\rangle$ into the basis $\langle x|$. That's a bit strange, but only because you only tend to see things being a function *of* x, for example. With Dirac's notation we can separate the abstract entity that is the function from the basis, x in this case. So we can write that $\langle x|c_n\rangle = \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right)$. This is useful because it allows us to write inner products of functions quite simply by using a trick called the 'resolution of the identity', which mathematically is

$$1 = \int_{-L}^{L} |x\rangle\langle x|.$$

All the stuff with functions in Dirac notation needs a domain to be defined on x, to set the limits of such integrals. That's OK, though, as you need to specify your domain in Fourier analysis anyway. Things like $|u\rangle\langle v|$ are sometimes called 'outer products'. If $|u\rangle$ and $|v\rangle$ are n component vectors, *i.e.* $1 \times n$ component matrices, then their inner product is a scalar (1×1 matrix) and their outer product is an $n \times n$ matrix.

The resolution of the identity isn't that strange a concept, really: as usual, just think about normal vectors. Say we're in $2D$, then a complete orthonormal basis is given by $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The resolution of the identity simply states that the sum of the 'outer products' of these states is the identity:

$$\sum_{i=1}^{2} |e_i\rangle\langle e_i| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is true for any complete orthonormal basis, including $|x\rangle$ above. The usefulness of this trick is as follows: for functions the inner product is

$$\langle u|v\rangle = \int_{-L}^{L} \langle u|x\rangle\langle x|v\rangle \mathrm{d}x = \int_{-L}^{L} u(x)^* \, v(x) \, \mathrm{d}x$$

where we inserted a 1 in the second step and used that $1 = \int_{-L}^{L} |x\rangle\langle x|$.

Now, say we choose to define

$$\begin{aligned}
\langle x|c_n\rangle &= \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right) \\
\langle x|s_n\rangle &= \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right) \\
\langle x|1\rangle &= \frac{1}{\sqrt{2L}}.
\end{aligned}$$

It's a bit strange at first thinking of functions as infinite dimensional vectors. To understand this, think of how a computer program would store a function. Computers do everything discretely with

arrays, so when you ask a plotting program to plot a continuous function over some domain of x it's actually got a large number of discrete x values, probably so densely packed as to look continuous. For each it has a y value, and again if everything's set up correctly the density of the points in x will be large enough that the function looks continuous in y, even if it's steeply sloped. So when you draw a function, by assigning a point in y to each point along your x axis, it would be indistingushable by eye if you had a very large set of (x, y) densely packed along x and if you had an infinite number. For more on this look up 'linear vector spaces' on wikipedia or in a textbook. It's quite straightforward - there are a list of 8 or so rules defining if some object is a vector, and it turns out that functions, correctly defined, obey the rules. In the computer example it's as if we've 'projected' the function onto the point x for each of a discrete set of points, hence why we write $\langle x|f\rangle = f(x)$.
The point of this digression is that we can show

$$\begin{aligned} \langle 1|c_n\rangle &= \langle 1|s_n\rangle = \langle c_n|s_m\rangle = 0 \\ \langle c_m|c_n\rangle &= \langle s_m|s_n\rangle = \delta_{nm} \end{aligned} \tag{1}$$

where the 'Kronecker delta' is defined as $\delta_{nm} = 1$ if $n = m$, 0 otherwise. The proofs are in the appendix, and you should check them, as they form the crux of the argument (and demonstrate the usual tricks for Fourier analysis). For now let's get back to Fourier.

So say we have a function we'd like to decompose into our orthonormal basis

$$|f\rangle = \frac{a_0}{2}\sqrt{2L}|1\rangle + \sum_{n=1}^{\infty} a_n\sqrt{L}|c_n\rangle + \sum_{n=1}^{\infty} b_n\sqrt{L}|s_n\rangle$$

which is just the first equation written in Dirac notation (the $\sqrt{2L}$, $\sqrt{L}$ come from the definitions of $|1\rangle = \frac{1}{\sqrt{2L}}$ *etc.* - they just ensure that the basis vectors are normalized, *i.e.* $\langle 1|1\rangle = 1$). How would you do it for vectors? Well, we could dot the whole thing with a vector which is perpendicular to two of the three vectors. So for example if we'd like the a_n coefficients we can take the inner product of $|f\rangle$ with the $|c_m\rangle$ basis function:

$$\langle c_m|f\rangle = \frac{a_0}{2}\sqrt{2L}\langle c_m|1\rangle + \sum_{n=1}^{\infty} a_n\sqrt{L}\langle c_m|c_n\rangle + \sum_{n=1}^{\infty} b_n\sqrt{L}\langle c_m|s_n\rangle$$

note that we used $\langle c_m|$ rather than $\langle c_n|$ in order that we could legitimately pull it through the sums. Just think of this as 'dotting from the left with the vector $\vec{c}_m$' - after all, that's what it is, but we've generalised our idea of 'dot' and 'vector'. Using Equations 1 this becomes

$$\langle c_m|f\rangle = \sqrt{L}\sum_{n=1}^{\infty} a_n\delta_{mn} = \sqrt{L}a_m$$

where the δ_{mn} selects out only the term $m = n$ from the sum. Rewriting back into the usual function notation , by inserting an identity as above, this equation states that

$$\frac{1}{\sqrt{L}}\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right) f(x)\,\mathrm{d}x = \sqrt{L}a_m$$

or that

$$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right) f(x)\,\mathrm{d}x = a_m$$

which is hopfeully what you have in your notes. To find the b_n term is just as simple:

$$\begin{aligned}
|f\rangle &= \frac{a_0}{2}\sqrt{L}|1\rangle + \sum_{n=1}^{\infty} a_n\sqrt{L}|c_n\rangle + \sum_{n=1}^{\infty} b_n\sqrt{L}|s_n\rangle \\
&\downarrow \quad \langle s_m| \rightarrow \\
\langle s_m|f\rangle &= \frac{a_0}{2}\sqrt{L}\langle s_m|1\rangle + \sum_{n=1}^{\infty} a_n\sqrt{L}\langle s_m|c_n\rangle + \sum_{n=1}^{\infty} b_n\sqrt{L}\langle s_m|s_n\rangle \\
&= \sum_{n=1}^{\infty} b_n\sqrt{L}\delta_{mn} \\
&= \sqrt{L}b_m
\end{aligned}$$

or, rewritten,

$$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)f(x)\,\mathrm{d}x = b_m$$

and finally

$$\begin{aligned}
\langle 1|f\rangle &= \frac{a_0}{2}\sqrt{2L} \\
\frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x &= a_0.
\end{aligned}$$

If the Dirac notation is a bit confusing try redoing all the above stuff just writing everything as functions. It all works exactly the same. I prefer the Dirac way because it lets you see that you already know how to do Fourier analysis - it's just decomposing vectors into basis vectors. It also helps when you get to quantum mechanics in your second year; it appears to be completely new, but quantum mechanics is again just manipulation of linear vector spaces.

Appendix

Proof that

$$\begin{aligned}
\langle x|c_n\rangle &= \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right) \\
\langle x|s_n\rangle &= \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right) \\
\langle x|1\rangle &= \frac{1}{\sqrt{2L}}
\end{aligned}$$

form an orthonormal basis over $x \in [-L, L)$, *i.e.* that

$$\begin{aligned}
\langle 1|c_n\rangle \quad &= \quad \langle 1|s_n\rangle = \langle c_n|s_m\rangle = 0 \\
\langle c_m|c_n\rangle \quad &= \langle s_m|s_n\rangle \quad = \delta_{nm}.
\end{aligned}$$

Let's check them explicitly. Any I miss out you can check yourself if you're not convinced.

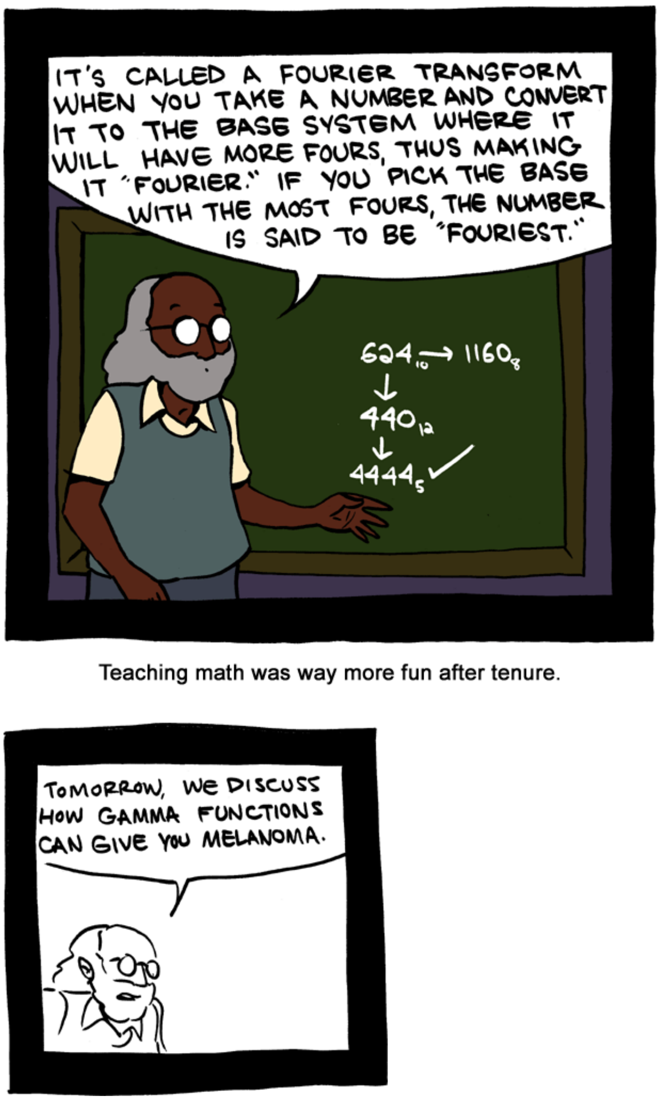

Figure 1: SMBC's briefer explanation of Fourier Analysis. Well, Fourier transforms - you'll encounter these later, but they can be considered a certain limit of Fourier series. Retrieved from http://www.smbc-comics.com/index.php?db=comics&id=2874

$$\begin{aligned}\langle 1|c_n\rangle &= \langle 1|\left(\int_{-L}^{L}|x\rangle\langle x|\mathrm{d}x\right)|c_n\rangle = \int_{-L}^{L}\langle 1|x\rangle\langle x|c_n\rangle \mathrm{d}x = \int_{-L}^{L}\frac{1}{\sqrt{2L}}\frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ &= \frac{1}{\sqrt{2}L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\ &= \frac{2}{\sqrt{2}n\pi}\left[\sin\left(n\pi\right)\right] = 0\end{aligned}$$

(there's no singularity to deal with since $n > 0$).

$$\langle c_n|s_m\rangle = \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

Note that this is the one I attempted to prove was δ_{mn} in the class - it's not! We found in the class that it was zero, and indeed it should be. Sorry about that. Use

$$\begin{aligned}\sin\left(A+B\right) &= \sin\left(A\right)\cos\left(B\right)+\cos\left(A\right)\sin\left(B\right) \\ \sin\left(A-B\right) &= \sin\left(A\right)\cos\left(B\right)-\cos\left(A\right)\sin\left(B\right)\end{aligned}$$

to rewrite as

$$\begin{aligned}\langle c_n|s_m\rangle &= \frac{1}{2L}\int_{-L}^{L}\left(\sin\left(\frac{(n+m)\pi x}{L}\right)-\sin\left(\frac{(n-m)\pi x}{L}\right)\right)\mathrm{d}x \\ &= \frac{-1}{2\pi}\left[\frac{1}{n+m}\cos\left(\frac{(n+m)\pi x}{L}\right)-\frac{1}{n-m}\cos\left(\frac{(n-m)\pi x}{L}\right)\right]_{-L}^{L} \\ &= 0\end{aligned}$$

where the last line follows from the fact that cos is an even function, and there is no problem when $n = m$ as we showed with l'Hôpital's rule in the class.

$$\langle c_m|c_n\rangle = \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

this time use

$$\begin{aligned}\cos\left(A+B\right) &= \cos\left(A\right)\cos\left(B\right)-\sin\left(A\right)\sin\left(B\right) \\ \cos\left(A-B\right) &= \cos\left(A\right)\cos\left(B\right)+\sin\left(A\right)\sin\left(B\right)\end{aligned}$$

to rewrite as

$$\begin{aligned}\langle c_m|c_n\rangle &= \frac{1}{2L}\int_{-L}^{L}\left(\cos\left(\frac{(m+n)\pi x}{L}\right)+\cos\left(\frac{(m-n)\pi x}{L}\right)\right)\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{1}{m+n}\sin\left(\left(m+n\right)\pi\right)+\frac{1}{m-n}\sin\left(\left(m-n\right)\pi\right)\right].\end{aligned}$$

The first term disappears without incident since the denominator cannot become zero. For the second term, check the two cases:

$$m \neq n: \langle c_m | c_n \rangle = \frac{1}{\pi} \frac{1}{m-n} \sin\left((m-n)\pi\right) = 0$$

and using l'Hôpital's rule for the other case, differentiating top and bottom with respect to $m-n$,

$$m = n: \langle c_m | c_n \rangle = \frac{\pi}{\pi} \frac{\cos\left((m-n)\pi\right)}{1} = 1$$

so we have proven that

$$\langle c_m | c_n \rangle = \delta_{mn}.$$