Faddeev Popov Ghosts

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These notes are to accompany my talk on gauge fixing in quantum field theories (QFTs), and the mathematical tricks associated with this procedure. In Section 1 I will look at gauge fixing for Abelian fields, considering the specific case of the Maxwell theory describing photons. In Section 2 I consider a non-Abelian generalization of the Maxwell theory called Yang Mills theory, which describes vector bosons or gluons. I demonstrate the necessity of introducing 'ghost' fields in order to fix the gauge. In Section 3 I look at how ghosts arise as a natural consequence of the geometry of QFTs.

1 Abelian fields

The Maxwell action is

$$S\left[A\right]=\int \mathrm{d}^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{1}$$

where the field strength ('curvature' in Maths language) is defined to be

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{2}$$

Physical observables in QFT are given by the vacuum expectation values of gauge invariant quantities $\mathcal{O}\left[A\right]$:

$$\langle\Omega|T\mathcal{O}\left[A\right]|\Omega\rangle=\frac{\int\mathscr{D}A\mathcal{O}\left[A\right]\exp\left(iS\left[A\right]\right)}{\int\mathscr{D}A\exp\left(iS\left[A\right]\right)} \tag{3}$$

where vacuum states (containing no particles) are denoted $|\Omega\rangle$, and T is the time-ordering operator (which is not important for what follows).

Just as in the case of classical electromagnetism, the field A is only defined up to a choice of gauge. That is, physical results should remain unchanged under a transformation

$$A^\alpha_\mu\left(x\right)=A_\mu\left(x\right)+\frac{1}{e}\partial_\mu\alpha\left(x\right) \tag{4}$$

with $\alpha\left(x\right)$ a Lorentz scalar field. This is the Lorentz covariant version of the classical statement that the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ is unaffected if we change the magnetic vector potential $\mathbf{A}$ to $\mathbf{A}^\alpha=\mathbf{A}+\frac{1}{e}\nabla\alpha\left(x\right)$. The field strength $F_{\mu\nu}$ in QED acts a little like the magnetic field in classical electrodynamics, and we can see that it is indeed invariant to the gauge choice.

The problem with having the gauge freedom of Equation 4 is that the functional integration $\mathscr{D}A$ has an infinite degeneracy built into it: for each physically unique field configuration there are an infinite number of equivalent configurations given by the choice of gauge. This situation is shown schematically in Figure 1.

Physically equivalent fields are said to lie on the same 'gauge orbit'. We need to pick one field from each gauge orbit in order to remove the degeneracy in our functional integral. If the space of all fields is denoted $\mathscr{A}$, we'd like to consider a subset $\mathscr{S}\subset\mathscr{A}$ which contains only physically inequivalent fields. This situation is given schematically in Figure 2.

First we define a functional $G\left[A\right]$ such that enforcing $G\left[A\right]=0$ fixes the gauge. In this talk I will use the Feynman 't Hooft Landau gauge, given by

$$G\left[A\right]=\partial^\mu A_\mu\left(x\right)+w\left(x\right). \tag{5}$$

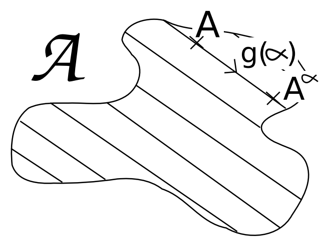


Figure 1: The space of all field configurations, $\mathscr{A}$, includes an infinite degeneracy of physically equivalent fields related by a choice of gauge. If two fields are related by a gauge transformation $g(\alpha)$ we say they sit on the same 'gauge orbit'. In this picture $\mathscr{A}$ has been drawn as a set of gauge orbits.

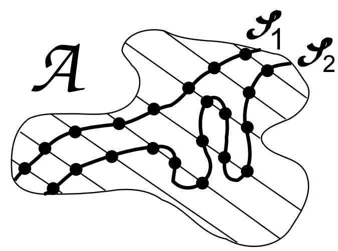


Figure 2: Gauge choices which select out a subset of all fields $\mathscr{S} \subset \mathscr{A}$. The choice $\mathscr{S}_1$ shows a complete gauge choice; examples include the Feynman 't Hooft Landau gauge $G = \partial^{\mu} A_{\mu}(x) + w(x)$. The choice $\mathscr{S}_2$ shows an incomplete gauge choice, in which there is a residual gauge freedom remaining. Examples of this type are the axial gauge $G = A_0(x)$ and the Coulomb gauge $G = \nabla_i A^i(x)$ (with $i \in [1,3]$).

Now we employ what I'll call the *first Faddeev Popov trick* (FP1):

$$\boxed{1 = \int \mathscr{D}\alpha \delta\left(G\left[A^{\alpha}\right]\right) \det\left(\frac{\delta G\left[A^{\alpha}\right]}{\delta \alpha}\right)} \tag{6}$$

where the delta function is understood to be infinite dimensional (a *delta functional,* perhaps). Although I can't prove Equation 6 I can give a plausibility argument for it: FP1 is the continuum limit of the N-dimensional equation

$$1 = \left(\prod_{i=1}^{N} \int \mathrm{d}\alpha_i\right) \delta^{(N)}\left(G_i\left(\alpha_j\right)\right) \det\left(\frac{\partial G_i}{\partial \alpha_j}\right), \qquad i, j \in [1, N] \tag{7}$$

and, if we take the simplest case of $N = 1$, the equation takes the trivial form

$$1 = \int \mathrm{d}\alpha \delta\left(G(\alpha)\right) \frac{\mathrm{d}G}{\mathrm{d}\alpha}. \tag{8}$$

In order to evaluate Equation 3 it is only necessary to evaluate the numerator, then set $\mathcal{O} = 1$ for the denominator. Applying FP1 we have

$$\int_{\mathscr{S}} \mathscr{D}A \mathcal{O}[A] \exp\left(iS[A]\right) = \int_{\mathscr{S}} \mathscr{D}A \int \mathscr{D}\alpha \mathcal{O}[A] \delta\left(G\left[A^{\alpha}\right]\right) \det\left(\frac{\delta G\left[A^{\alpha}\right]}{\delta \alpha}\right) \exp\left(iS[A]\right). \tag{9}$$

The determinant can be found explicitly given our choice of gauge (Equation 5):

$$\begin{aligned} G\left[A^{\alpha}\right] &= \partial^{\mu} A_{\mu} + \frac{1}{e} \partial^{\mu} \partial_{\mu} \alpha + w \\ &\Downarrow \\ \frac{\delta G}{\delta \alpha} &= \frac{1}{e} \Box \end{aligned}$$

where $\Box \triangleq \partial^\mu \partial_\mu$ is the d'Alembertian. The important point to note is that the determinant (whatever it is) is independent of both α and of A, so can be pulled out of the functional integrals to give a constant prefactor. Thus we have that

$$\int_{\mathscr{S}} \mathscr{D}A\mathcal{O}[A]\exp(iS[A]) = \det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right)\int_{\mathscr{S}} \mathscr{D}A\int \mathscr{D}\alpha\mathcal{O}[A]\,\delta(G[A^\alpha])\exp(iS[A]). \tag{10}$$

Furthermore, noting that both $S[A]$ and $\mathcal{O}[A]$ are gauge invariant, we can replace them with $S[A^\alpha]$ and $\mathcal{O}[A^\alpha]$ respectively. The integration measure is also unchanged by gauge transformations (as they're just linear shifts), so we have

$$\begin{aligned}\int_{\mathscr{S}} \mathscr{D}A\mathcal{O}[A]\exp(iS[A]) &= \det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right)\int_{\mathscr{S}} \mathscr{D}A^\alpha\mathscr{D}\alpha\mathcal{O}[A^\alpha]\,\delta(G[A^\alpha])\exp(iS[A^\alpha]) \\ &= \det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right)\int \mathscr{D}\alpha\int_{\mathscr{S}} \mathscr{D}A\mathcal{O}[A]\,\delta(G[A])\exp(iS[A])\end{aligned} \tag{11}$$

where we have simply relabelled A^α to A in the second line (it's a dummy function!). The functional integral over α now gives an infinite constant, which will cancel out later on.

In Equation 11 we have achieved the desired result, in that the delta function enforces our gauge choice and restricts the functional integral from the space of all fields $\mathscr{A}$ to the space of all *physically ineqivalent* fields $\mathscr{S}$. Before finishing with the Abelian case, though, there's a further trick which makes the physics of the problem more apparent.

If we functionally integrate both sides of Equation 11 with a Gaussian distribution over w, $\int \mathscr{D}w\exp\left(-i\int \mathrm{d}^4x w^2/2\xi\right)$, the left hand side (being independent of w) is simply multiplied by a function of ξ. Bringing this across to the right, and grouping the various prefactors together, we have that

$$\begin{aligned}\int_{\mathscr{S}} \mathscr{D}A\mathcal{O}[A]\exp(iS[A]) &= \mathcal{N}(\xi)\int \mathscr{D}w\int_{\mathscr{S}} \mathscr{D}A\mathcal{O}[A]\,\delta(\partial^\mu A_\mu + w)\exp\left(-i\int \mathrm{d}^4x\frac{1}{2\xi}w^2\right)\exp(iS[A]) \\ &= \mathcal{N}(\xi)\int_{\mathscr{A}} \mathscr{D}A\mathcal{O}[A]\exp\left(-i\int \mathrm{d}^4x\frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)\exp(iS[A])\end{aligned} \tag{12}$$

or alternatively

$$\int_{\mathscr{S}} \mathscr{D}A\mathcal{O}[A]\exp(iS[A]) = \mathcal{N}(\xi)\int_{\mathscr{A}} \mathscr{D}A\mathcal{O}[A]\exp(iS_{GF}[A]) \tag{13}$$

with

$$S_{GF} = \int \mathrm{d}^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right) \tag{14}$$

the gauge fixed action. Finally we have the desired result:

$$\langle\Omega|T\mathcal{O}[A]|\Omega\rangle = \frac{\int_{\mathscr{A}} \mathscr{D}A\mathcal{O}[A]\exp(iS_{GF}[A])}{\int_{\mathscr{A}} \mathscr{D}A\exp(iS_{GF}[A])}. \tag{15}$$

1.1 The Maxwell propagator

Using integration by parts we can rewrite Equation 14 as

$$S_{GF} = \int \mathrm{d}^4x\frac{1}{2}A_\mu\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu + \frac{1}{\xi}\partial^\mu\partial^\nu\right)A_\nu \tag{16}$$

with $\eta^{\mu\nu}$ the Minkowski metric. In momentum space this is

$$S_{GF} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\tilde{A}_\mu\left(\eta^{\mu\nu}(-k^2) + k^\mu k^\nu - \frac{1}{\xi}k^\mu k^\nu\right)\tilde{A}_\nu. \tag{17}$$

The term sandwiched between the gauge fields is the inverse of the photon propagator, $(D^{\mu\nu})^{-1}$. The propagator itself is uniquely defined by this, and can easily be verified to be

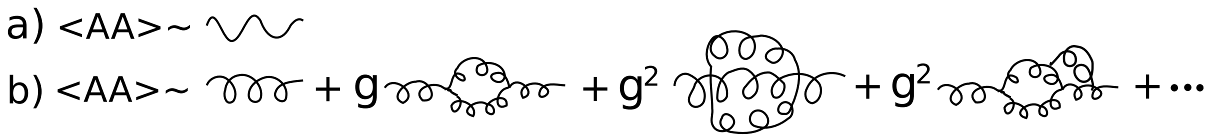


Figure 3: a) Photons have no self-interactions, so the tree-level propagator is exact. b) Gluons (and vector bosons) have 3-pt and 4-pt vertices, so there are loop contributions to the propagator.

$$D^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{18}$$

where the small imaginary term $i\epsilon$ has been added to make things better defined in calculations (again, it's not necessary in what follows). Note that if we hadn't applied the gauge fix, the last term in the inverse propagator in Equation 17 would not be present, and the inverse propagator would not have been invertible (so the propagator wouldn't exist).

The propagator features a term proportional to ξ, which was introduced in Equation 12 by functionally integrating the gauge fixing delta function with $\int \mathscr{D}w \exp\left(-i\int \mathrm{d}^4x w^2/2\xi\right)$. Taking the limit $\xi \to 0$ therefore rigidly enforces the gauge fixing condition. However, there's no requirement to do this in the quantum theory - after all, we introduced ξ arbitrarily. From the form of Equation 18 we see that ξ has something to do with regulating the transverseness of the photons' polarizations. If we choose to take $\xi \to 0$ all photons (even virtual photons[1]) must be transverse. If ξ takes a finite value, virtual photons are allowed to have a longitudinal component to their polarizations, with photon polarizations having a Gaussian distribution of longitudinal components centred on zero[2].

In the case of quantum electrodynamics (QED), which governs the coupling of photons to electrons, the unphysical nature of the longitudinal virtual photons is not a problem. All Feynman diagrams with contributions from longitudinal photons cancel one another, as a result of the *Ward identity*, which follows from applying Noether's theorem to the continuous symmetry associated with the invariance of $\mathscr{D}A$ to gauge transformations.

2 Non-Abelian fields

Yang Mills theory provides an extension of Maxwell theory to the case of non-Abelian fields. The action takes the following form:

$$S[A] = \mathrm{tr}\int \mathrm{d}^4x \left(-\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a\right) \tag{19}$$

where the field strength is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \tag{20}$$

The non-Abelian fields are now matrix valued: $A_\mu = A_\mu^a t_a$, with t_a the basis elements of the relevant group. For example, if our group is $SU(2)$ we could choose t_a to be the Pauli matrices. The trace would then be the trace of these 2×2 matrices. The structure constants are defined by $if^{abc}t_c \triangleq \left[t^a, t^b\right]$.

One major difference between Yang Mills theory and Maxwell theory, and consequently between gluons and photons, is that gluons have self-interactions, as can be seen from the terms $\sim A^2\partial A$ and A^4 in Equation 19. This makes things much more difficult. The situation is depicted in Figure 3.

Gauge transformations of the field A_μ now take the form

$$\begin{aligned}(A^\alpha)_\mu^a &= A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A_{\mu b}\alpha_c \\ &= A_\mu^a + \frac{1}{g}D_\mu^{ab}\alpha_b \end{aligned} \tag{21}$$

[1]Virtual photons appear as internal lines in Feynman diagrams. This requires an interacting theory, such as QED, rather than the free Maxwell theory. Still, the point remains.

[2]For interest, $\xi = 0$ is called the 'Landau gauge' and $\xi = 1$ the 'Feynman gauge'. I don't know where these names came from, but the choice of ξ (a Lagrange multiplier) is not a gauge choice, so they're misleading.

with

$$D_{\mu}^{ab} \triangleq \delta^{ab}\partial_{\mu} + gf^{abc}A_{c\mu}. \tag{22}$$

Equation 21 can be compared with Equation 4 of the Abelian case.

We proceed as before to evaluate the vacuum expectation values of gauge invariant operators. We again choose the Feynman 't Hooft Landau gauge:

$$G = \partial^{\mu}A_{\mu}^{a} + w^{a} \tag{23}$$

and we need to evaluate

$$\langle\Omega|T\mathcal{O}[A]|\Omega\rangle = \frac{\mathrm{tr}\int_{\mathcal{A}}\mathscr{D}A\mathcal{O}[A]\exp(iS[A])}{\mathrm{tr}\int_{\mathcal{A}}\mathscr{D}A\exp(iS[A])}. \tag{24}$$

Focussing on the numerator as before, we apply FP1 (Equation 6):

$$\mathrm{tr}\int_{\mathcal{A}}\mathscr{D}A\mathcal{O}[A]\exp(iS[A]) = \mathrm{tr}\int_{\mathcal{A}}\mathscr{D}A\int\mathscr{D}\alpha\delta(G[A^{\alpha}])\det\left(\frac{\delta G[A^{\alpha}]}{\delta\alpha}\right)\mathcal{O}[A]\exp(iS[A]). \tag{25}$$

In the Abelian case we now used the fact that the determinant is independent of A to pull it out of the functional integral. Let's see if this holds for the non-Abelian case.

$$\begin{aligned} G[A^{\alpha}] &= \partial^{\mu}A_{\mu}^{a} + \frac{1}{g}\partial^{\mu}D_{\mu}^{ab}\alpha_b + w^{a} \\ &\Downarrow \\ \frac{\delta G}{\delta\alpha} &= \frac{1}{g}\partial^{\mu}D_{\mu}^{ab} = \frac{1}{g}\Box + f^{abc}\partial^{\mu}A_{c\mu}. \end{aligned} \tag{26}$$

So the determinant in the non-Abelian case is a function of A, and we can't use the same simplification as before. It is however independent of α, meaning we can at least use the trick of changing the gauge invariant quantities $\mathscr{D}A$, $\mathcal{O}[A]$, and $S[A]$ to $\mathscr{D}A^{\alpha}$, $\mathcal{O}[A^{\alpha}]$, and $S[A^{\alpha}]$, and then relabelling $A^{\alpha}\to A$. Thus Equation 25 can be rewritten

$$\mathrm{tr}\int_{\mathcal{A}}\mathscr{D}A\mathcal{O}[A]\exp(iS[A]) = \left(\int\mathscr{D}\alpha\right)\mathrm{tr}\int_{\mathcal{A}}\mathscr{D}A\delta\left(\partial^{\mu}A_{\mu}^{a}+w^{a}\right)\det\left(\frac{1}{g}\partial^{\mu}D_{\mu}^{ab}\right)\mathcal{O}[A]\exp(iS[A]). \tag{27}$$

One more trick carrying over from the Abelian case is to functionally integrate both sides of the equation with $\int\mathscr{D}w\exp\left(-i\int \mathrm{d}^4x w^2/2\xi\right)$. Again the left hand side is just multiplied by a function of ξ, and the delta functional picks out the gauge fixing condition:

$$\mathrm{tr}\int_{\mathscr{S}}\mathscr{D}A\mathcal{O}[A]\exp(iS[A]) = \mathcal{N}(\xi)\,\mathrm{tr}\int_{\mathscr{A}}\mathscr{D}A\det\left(\frac{1}{g}\partial^{\mu}D_{\mu}^{ab}\right)\mathcal{O}[A]\exp(iS_{GF}[A]) \tag{28}$$

with

$$S_{GF} = \mathrm{tr}\int \mathrm{d}^4x\left(-\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^{a} - \frac{1}{2\xi}\left(\partial^{\mu}A_{\mu}^{a}\right)^2\right). \tag{29}$$

How do we deal with the determinant? The most elegant solution was again proposed by Faddeev and Popov, and is what I'll call the *second Faddeev Popov trick* (FP2):

$$\boxed{\det\left(\frac{1}{g}\partial^{\mu}D_{\mu}^{ab}\right) = \int\mathscr{D}\bar{c}\mathscr{D}c\exp\left(i\int \mathrm{d}^4x\bar{c}_a(x)\left(-\partial^{\mu}D_{\mu}^{ab}\right)c_b(x)\right)} \tag{30}$$

(the coupling g has been absorbed into the definition of c). The fields $c_a(x)$ and $\bar{c}_a(x)$ are complex anticommuting Lorentz scalar fields. To see where FP2 comes from, let's again consider the finite dimensional case, in which we have *Grassman variables* c_i and $\bar{c}_i$. Grassman variables for the case $i=1$ are just anticommuting complex numbers. If $i\in[1,N]$ then c_i is an N component complex vector satisfying these anticommutation relations:

$$\{c_i, c_j\} = \{\bar{c}_i, c_j\} = \{\bar{c}_i, \bar{c}_j\} = 0 \tag{31}$$

with $\bar{c}_i$ being the equivalent of the complex conjugate to c_i. Calculus is defined over this field by

$$\int \mathrm{d}c_i c_i = 1, \qquad \int \mathrm{d}c_i = 0 \tag{32}$$

with the equivalent for the $\bar{c}_i$s. We are interested in evaluating

$$\left(\prod_{i=1}^{N} \int \mathrm{d}\bar{c}_i\right)\left(\prod_{i=1}^{N} \int \mathrm{d}c_i\right) \exp\left(-\bar{c}_i A^{ij} c_j\right). \tag{33}$$

Note that the exponential, defined by its Taylor series, terminates at finite order due to the anticommutation relations. For $i \in [1, N]$ the series terminates at the N^{th} order. For simplicity we'll consider $N = 1$. The second term in the series is $\frac{1}{2}\bar{c}Ac\bar{c}Ac = -\frac{1}{2}\bar{c}A\bar{c}cAc = -\frac{1}{2}\bar{c}^2A^2c^2 = 0$, and similarly for higher terms (A is a commuting complex number here, or a commuting complex matrix in general). Therefore we have

$$\begin{aligned}
\int \mathrm{d}\bar{c}\mathrm{d}c \exp\left(-\bar{c}Ac\right) &= \int \mathrm{d}\bar{c}\mathrm{d}c \left(1 - \bar{c}Ac\right) \\
&= -A \int \mathrm{d}\bar{c} \int \mathrm{d}c\bar{c}c \\
&= A \int \mathrm{d}\bar{c}\bar{c} \int \mathrm{d}cc \\
&= A
\end{aligned} \tag{34}$$

which can be contrasted with the case for commuting complex numbers $z = x + iy$:

$$\begin{aligned}
\int \mathrm{d}\bar{z}\mathrm{d}z \exp\left(-\bar{z}Az\right) &= \int \mathrm{d}x\mathrm{d}y \exp\left(-\left(x^2 + y^2\right) A\right) \\
&= \frac{\pi}{A}
\end{aligned} \tag{35}$$

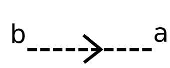
from which we see the necessity of using anticommuting rather than commuting fields.
Applying FP2 to our numerator (Equation 28) we have the final result

$$\operatorname{tr} \int_{\mathscr{S}} \mathscr{D}A\mathcal{O}[A] \exp\left(iS[A]\right) = \mathcal{N}(\xi) \operatorname{tr} \int_{\mathscr{A}} \mathscr{D}A \int \mathscr{D}\bar{c}\mathscr{D}c\mathcal{O}[A] \exp\left(iS_{FP}[A, \bar{c}, c]\right) \tag{36}$$

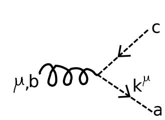
with

$$S_{FP} = \operatorname{tr} \int \mathrm{d}^4x \left(-\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a - \frac{1}{2\xi}\left(\partial^\mu A_\mu^a\right)^2 + \bar{c}_a\left(-\partial^\mu D_\mu^{ab}\right)c_b\right). \tag{37}$$

The final form of our action suggests that gauge fixing in the non-Abelian case requires the introduction of additional (anticommuting complex scalar) fields. These fields are called *Faddeev Popov ghosts*. Expanding the final term in Equation 37 we have both a ghost propagator $\langle\bar{c}c\rangle$ and an antighost-gluon-ghost vertex, with corresponding Feynman rules



$$= \frac{i\delta^{ab}}{k^2 + i\epsilon} \tag{38}$$



$$= -gf^{abc}k^\mu. \tag{39}$$

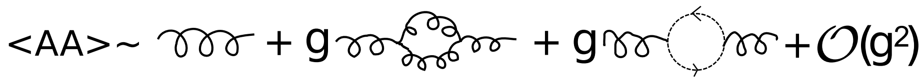


Figure 4: The 2-pt propagator for the gluon field, $\langle AA \rangle$, to order g (one loop). Gluons are given by coiled lines, and ghosts by dashed lines.

These rules add extra Feynman diagrams to those in Figure 3 b. The modified 2-pt function for gluons is given in Figure 4.

Note that the ghost field describes anticommuting spin-0 (scalar) particles. This is in contradiction to the spin statistics theorem, which states that fermionic particles (odd-half-integer spin) are antisymmetric under exchange (the fields anticommute) whereas bosonic particles (integer spin) are symmetric under exchange (the fields commute). It is a general property of ghosts in any QFT that they violate spin statistics. For this reason, and because they only appear as internal lines in Feynman diagrams, we say that ghosts are not 'real'. They are a mathematical trick required to patch up perturbation theory.

3 BRST symmetry

In Section 1 we saw that virtual photons can have unphysical longitudinal polarizations, but that the contributions to scattering amplitdes from these photons exactly cancel out. This is a result of the Ward identity (or the Ward Takahashi identity if the photons are also allowed to move off-shell). Do we have a similar scenario for the non-Abelian case? The action (Equation 37) appears to allow unphysical longitudinal gluons. It turns out that the Ward identity no longer cancels these contributions, but a new symmetry does. We call this BRST (Becchi Rouet Stora Tyutin) symmetry, and it arises from the geometry of gauge fields.

Rewriting the Lagrangian density slightly we have

$$\mathscr{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \bar{c}_a\left(-\partial^\mu D^{ab}_\mu\right)c_b \tag{40}$$

which we can rewrite by introducing a scalar commuting *auxiliary field* B^a like so:

$$\mathscr{L}' = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{\xi}{2}\left(B^a\right)^2 + B^a\partial^\mu A_{a\mu} + \bar{c}_a\left(-\partial^\mu D^{ab}_\mu\right)c_b \tag{41}$$

where the equality follows from completing the square. The remaining term disappears upon functional integration over the B field:

$$\int \mathscr{D}A\mathscr{D}\bar{c}\mathscr{D}c\exp\left(i\int \mathrm{d}^4x\mathscr{L}\left(A,\bar{c},c\right)\right) = \left(\frac{\xi}{2\pi}\right)^2\int \mathscr{D}A\mathscr{D}\bar{c}\mathscr{D}c\mathscr{D}B\exp\left(i\int \mathrm{d}^4x\mathscr{L}'\left(A,\bar{c},c,B\right)\right). \tag{42}$$

BRST is an additional symmetry of $\mathscr{L}'$, not present in the bare Yang Mills Lagrangian, which comes about upon the introduction of the ghosts and the auxiliary field. For an infinitesimal perturbation ϵ, with ϵ a Grassman variable, we have that $\delta\mathscr{L}' = 0$ for

$$\begin{aligned}
\delta A^a_\mu &= \epsilon D^{ac}_\mu c^c \\
\delta c^a &= -\frac{1}{2}g\epsilon f^{abc}c_b c_c \\
\delta\bar{c}^a &= \epsilon B^a \\
\delta B^a &= 0.
\end{aligned} \tag{43}$$

The first term in Equation 43 is simply a local gauge transformation with parameter $\alpha^a(x) = g\epsilon c^a(x)$, which means that the first and second terms in Equation 41 are trivially invariant. With some rearrangement the remaining variation is found to be $\delta\mathscr{L}' = -\frac{1}{2}\epsilon g^2 f^{abc}f_{cde}\left(A_{b\mu}c^d c^e + A^d_\mu c^e c_b + A^e_\mu c_b c^d\right)$, which vanishes because of the Jacobi identity.

Defining the BRST operator Q such that $\delta\phi = \epsilon Q\phi$ (for ϕ any of the fields), the geometrical nature of the symmetry begins to take shape. First we note that applying the operator twice (the variation of the variation) gives zero, showing that Q is 'nilpotent':

$$Q^2 = 0. \tag{44}$$

This property holds for the boundary operator ∂ in algebraic geometry, which maps (for example) a manifold $\mathcal{M}$ to its boundary $\partial\mathcal{M}$. Nilpotent operators have the property that they split up the space on which they act into three distinct subspaces. To make this statement precise, let's consider the canonically quantized Yang Mills theory (with fermions, ghosts, and auxiliary fields). The continuous BRST symmetry must have a conserved charge Q (the eigenvalues of the Q operator) by Noether's theorem. Being conserved we know the operator Q commutes with the Hamiltonian:

$$\dot{Q} = 0, \qquad \therefore \quad [Q, H] = 0. \tag{45}$$

Our Hilbert space $\mathcal{H}$ is now divided into three subspaces by Q. We can write it as a direct sum of these subspaces $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, where

$$\begin{array}{lll} |\psi_1\rangle \in \mathcal{H}_1, & \text{such that} & Q|\psi_1\rangle \neq 0 \\ |\psi_2\rangle \in \mathcal{H}_2, & \text{such that} & |\psi_2\rangle = Q|\psi_1\rangle \quad (\therefore Q|\psi_2\rangle = 0) \\ |\psi_0\rangle \in \mathcal{H}_0, & \text{such that} & Q|\psi_0\rangle = 0, \text{ but } |\psi_0\rangle \neq Q|\psi_1\rangle. \end{array} \tag{46}$$

In Maths-speak, we say that the vector space annihilated by Q is its *kernel*: $\mathcal{H}_0 \oplus \mathcal{H}_2 = \ker(Q)$, whereas the vector space comprising vectors of the form $Q|\psi\rangle$ is the *image* of Q: $\mathcal{H}_2 = \mathrm{im}\,(Q)$.

To see the relevance of this, consider the limit of zero coupling $g \to 0$, in which we approach the Abelian case. In this case, Q acts as follows: it annihilates single ghosts, it converts forward-polarized components of A to ghosts, and it converts antighosts to backward-polarized components of A[3]. In terms of our Hilbert space decompostion, then, we see that forward-polarized components of A and antighosts both live in $\mathcal{H}_1$, backward-polarized components of A and ghosts live in $\mathcal{H}_2$, and the transverse components of A (the only physical bits of the whole theory) live in $\mathcal{H}_0$. Thus BRST symmetry shows that the unphysical polarization states of non-Abelian gauge theories are exactly cancelled by the Faddeev Popov ghosts.

Let's look at this statement a different way. Again in the $g \to 0$ limit, choosing the Feynman 'gauge' $\xi = 1$, and working in d dimensions, our partition function is

$$\begin{aligned} \mathscr{Z} &= \int \mathscr{D}A\mathscr{D}\bar{c}\mathscr{D}c \exp\left(i\int \mathrm{d}^d x \left(-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{c}_a\left(-\partial^\mu D^{ab}_\mu\right)c_b\right)\right) \\ &= \int \mathscr{D}A \exp\left(i\int \mathrm{d}^d x \frac{1}{2} A_\mu \Box \eta^{\mu\nu} A_\nu\right) \int \mathscr{D}\bar{c}\mathscr{D}c \exp\left(i\int \mathrm{d}^d x \bar{c}_a\left(-\Box\delta^{ab}\right)c_b\right) \\ &= \left(\det\left(-\Box\right)\right)^{-d/2} \cdot \left(\det\left(-\Box\right)\right)^{+2/2}. \end{aligned} \tag{47}$$

This tells us that the ghosts act as 'negative degrees of freedom', which cancel out the unphysical degrees of freedom ($\sim$unphysical polarization states) afforded us by the ambiguity in gauge. By construction we see that the ghosts cancel exactly two degrees of freedom, which is intuitively correct in $d = 4$: we should have 2 transverse polarizations, but A_μ is a 4-vector, with 4 degrees of freedom. The ghosts cancel the other two.

3.1 The S-matrix

As a final remark on BRST symmetry, I will prove that the S-matrix projected onto the physical $\mathcal{H}_0$ subspace is unitary, meaning that we can work entirely in that subspace rather than in the whole Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ (in which S is automatically unitary).

From the definitions of the states $|\psi_i\rangle$ in Equation 46, and the Hermiticity of Q, it follows that the inner product of any two states in $\mathcal{H}_2$ is zero:

$$\langle\psi_2^A|\psi_2^B\rangle = \langle\psi_2^A|Q|\psi_1^B\rangle = \left(Q|\psi_2^A\rangle\right)^\dagger |\psi_1^B\rangle = 0 \tag{48}$$

and similarly the product of any state in $\mathcal{H}_2$ with a state in $\mathcal{H}_0$ is also zero:

$$\langle\psi_2^A|\psi_0^B\rangle = 0. \tag{49}$$

We need to show that S projected onto $\mathcal{H}_0$ is unitary, *i.e.*

[3]Directly it converts $\bar{c} \to B$, but varying the action with respect to B to get the equations of motion, we find that $\xi B^a = -\partial^\mu A^a_\mu$, showing that Q in fact acts to convert antighosts to backward-polarized components of A.

$$\sum_C \langle\psi_0^A|S^\dagger|\psi_0^C\rangle\langle\psi_0^C|S|\psi_0^B\rangle \equiv \langle\psi_0^A|\mathbb{I}|\psi_0^B\rangle \tag{50}$$

with $\mathbb{I}$ the identity. We can choose our initial state to be $|\psi_{in}\rangle = |\psi_0^A\rangle \in \mathcal{H}_0$. Now, since Q commutes with the Hamiltonian, we know that if the initial state is an eigenvector of Q with eigenvalue zero then the final (time evolved) state must also be an eigenvector of Q with eigenvalue zero (as Q and H are simultaneously diagonalizable, and the total S matrix is unitary). Therefore we have that

$$|\psi_{out}\rangle = S|\psi_0^A\rangle \in \mathcal{H}_0 \oplus \mathcal{H}_2. \tag{51}$$

We already know that the inner product of any $|\psi_2\rangle$ state with either a $|\psi_2\rangle$ or a $|\psi_0\rangle$ state is zero, so any contribution to the inner product of two final states can only come from the inner product of $|\psi_0\rangle$ states. That is,

$$\left(\langle\psi_0^A|S^\dagger\right)\left(S|\psi_0^B\rangle\right) = \sum_C \left(\langle\psi_0^A|S^\dagger\right)|\psi_0^C\rangle\langle\psi_0^C|\left(S|\psi_0^B\rangle\right) \tag{52}$$

which is exactly Equation 50.

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