A Practical Introduction to Quantum Field Theory

Felix Flicker

0.1 A Practical Introduction

The aim of these lectures is to provide a very brief introduction to the philosophy and some of the key tools of quantum field theory (QFT). Inevitably with such a short course a huge amount will be omitted. To minimise the damage I will try to present a technical introduction to some of the methods used day-to-day in practical field theory, and will discuss the philosophy of the subject around the mathematics.

There are two reasons for this choice. First, QFT is fundamentally a practical, pragmatic subject. It has unmatched predictive power, and typifies the modern way we understand the universe - but at the same time it is riddled with philosophical and mathematical issues. In many cases we don't know *why* it works, it just does[1]. Second, there are many aspects of field theory which are necessary for a full understanding but which aren't used day-to-day in research. An illustrative example is the quantization of classical fields. When calculating scattering amplitudes, for example, particles are described by quantum fields from the outset. Starting from the familiar idea of classical fields and turning them into the more complicated quantum fields may help build intuition, and has its place in a longer course - but I have rarely seen it used outside the field of quantum gravity, and it will not make it into this 'practical' guide.

One topic which is notably missing from this guide is second quantization. This is important to the philosophy of QFT and is also routinely used in research. I have found it possible to tell a self-contained story without it, and refer the interested reader to the literature.

0.2 Note on Conventions

My convention is to use $\triangleq$ to mean 'equal to by definition'. The symbol $\equiv$ is reserved for 'is equivalent to', in the sense that $\cos^2(\theta) + \sin^2(\theta) \equiv 1$, whereas $\cos(\theta) = 1$ could be a specific solution to something (but $\cos(\theta)$ and 1 are not equivalent).

I will assume Einstein summation notation throughout. I will make some limited effort to ascribe covariant quantities lower indices and contravariant upper, but since they can simply be raised and lowered with a metric I may well forget to do so; please interpret ∂^μ as $\eta^{\mu\nu}\partial_\nu$. Should it come up I'll use the metric $(+---)$.

Lower case symbols denote 4-vectors, *e.g.* $x_\mu = (t, \mathbf{x})_\mu$. The inner (dot) product behaves as is appropriate for the things it's dotting, so $x \cdot p \equiv x_\mu p^\mu$ and $\mathbf{x} \cdot \mathbf{p} \equiv \mathbf{x}_i \mathbf{p}^i$. I will also define $p^2 \triangleq p \cdot p$ (*etc.*) and the d'Alembertian $\Box \triangleq \partial^2 = \partial_t^2 - \nabla^2 = \partial_t^2 - \triangle$.

Fourier transforms and their inverses are defined as:

$$\begin{aligned} \tilde{\varphi}(p) &= \int \mathrm{d}^4 x \exp(ip \cdot x)\, \varphi(x) \\ \varphi(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-ip \cdot x)\, \tilde{\varphi}(p) \end{aligned}$$

where I will stick to the convention of writing twiddles over transformed quantities (in the literature both fields are usually given the same name).

0.3 Suggested Textbooks

There was until recently a famous absence of QFT textbooks aimed at people new to the subject. I believe this changed last year with the release of Tom Lancaster and Steve Blundell's *Quantum Field Theory for the Gifted*

[1] See, for example, Haag's theorem, which proves you cannot even consistently define the *vacuum* in an interacting field theory! For mathematical issues see the many failed attempts to create an Axiomatic QFT.

Amateur [3]. It covers a wide range of material very clearly and can be considered the course textbook (although these notes do not follow it directly).

The classic QFT text is Peskin and Schroeder [2], which is amusingly called an Introduction but which goes further than any course I've so far encountered. Although general, it leans slightly towards particle physics. For the more condensed matter-minded there's the equally brilliant Altland and Simons [4]. I found Zee [5] a nice read when I first met QFT, but it's probably best approached as a technical popular science book rather than a textbook. Think of it as the QFT version of the Feynman lectures - if you read it all those technical details will fit into place, but you're probably not going to learn the subject from it first time around.

The most relevant resource for this course, however, is provided by the notes I was given as an undergraduate. That course was prepared by John Chalker, a condensed matter theorist, and André Lukas, a high energy physicist and string theorist. The notes for their course are truly excellent, and I provide a link to them on my website. They are unmatched in their clarity and conciseness: almost all of the material in the notes you are reading is covered in Section 1.1 of Chalker and Lukas!

I will mention at the start of each section particularly relevant reading for the coming topics, taken from the References below.

References

[1] J. T. Chalker and A. Lukas, Theoretical Physics Lecture Notes (Oxford University), link available on my website (link)

[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Perseus Books, 1995

[3] T. Lancaster and S. J. Blundell, *Quantum Field Theory for the Gifted Amateur*, Oxford University Press, 2014

[4] A. Altland and B. Simons, *Condensed Matter Field Theory*, Cambridge University Press, 2006

[5] A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, 2010

[6] R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, available on Michael Berry's website (link)

Acknowledgments

First and foremost I would like to thank my entire cohort of friends from the Perimeter Institute's PSI programme for bringing their expertise from all areas of physics to these notes. Particular thanks are owed to Bruno Amorim, Eduardo Tremea Casali, Lauren Greenspan, Sebastian Montes Valencia, James Reid, and Nick Jones for thorough readings of the first draft with many helpful comments.

1 Actions and Lagrangians

The working of this section, and in fact Sections 2 and 3, follows very closely the working presented to me in my first course on QFT in [1]. This first chapter additionally takes a lot of working from Peskin and Schroeder, probably *the* classic QFT textbook (mainly written from a particle physics point of view) [2].

1.1 Natural Units

You will rarely see $\hbar$ or c appear in a modern field theory paper. We say both are 'set to one'. In fact we just choose sensible units. In relativistic systems it becomes apparent that space and time are really the same thing, so should be measured in the same units. This does away with c. In a similar manner all fundamental constants gradually disappear as we realize energies are just inverse times ($\hbar$), which are also temperatures (k_B), *etc. etc.* - in the end all units are simply comparisons to scales humans find accessible. They do help by providing added redundancy in calculations to check you've not made a mistake, but at some point more errors are introduced by trying to keep track of them.

It is convenient to leave one dimension in. A typical choice of dimension is Energy, $\mathbb{E}$, and a typical choice of unit is GeV. When asked for the dimension of some mathematical object we can state a number corresponding to the power of Energy. For example, $[Energy] = 1$, because the dimension of energy is $\mathbb{E}^1$. Similarly, $[Time] = \mathbb{E}^{-1}$ so $[Time] = -1$, and $[Length] = -1$, $[k_B] = [\hbar] = [c] = 0$, $[m] = [p] = [\partial^\mu] = 1$ *etc.* These 'natural units' are are what I will employ throughout this course. The argument goes that you can always 'put in the units again later' - well, you *could* convert them, or you *could* not bother - but in the latter case you might need to state a length in GeV^{-1}. In reality you probably won't be putting many numbers in anyway so the issue rarely comes up, and the algebra is easier with no constants kicking about.

1.2 Examples of Actions

The key object in a QFT is the action. This completely defines the theory of interest. Some examples from real theories are:

$$\begin{aligned}
S_{\text{Klein-Gordon}}[\varphi] &= \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + j\varphi\right) \\
S_{\text{Maxwell}}[A_\mu] &= \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu\right), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\
S_{\text{Schrödinger}}[\psi] &= \int \mathrm{d}^3\mathbf{x}\mathrm{d}t \left(\psi^\dagger\left(i\partial_t - H\right)\psi + j\psi^\dagger + j^\dagger\psi\right) \\
S_{\text{Chern-Simons}}[A_\mu] &= \int \mathrm{d}^3x \left(\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho + j_\mu A^\mu\right) \\
S_{\text{Klein-Gordon}}[\tilde{\varphi}] &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{1}{2}\tilde{\varphi}\left(p^2 - m^2\right)\tilde{\varphi} + \tilde{j}\tilde{\varphi}\right).
\end{aligned} \tag{1}$$

Things to note:

- the Schrödinger action is not relativistically covariant. That's okay, we use it in condensed matter where the presence of a lattice breaks Lorentz symmetry. I'll mainly consider general theories rather than specializing to condensed matter, particle physics, cosmology and so on - field theory is ubiquitous.
- The actions are *functionals*, *i.e.* functions of functions. This is symbolised by the brackets in $S[\varphi]$. We'll see some examples of functionals shortly.
- The fields themselves are functions. This could be a function of spacetime like $\varphi(x_\mu)$, a function of space and time separately like $\psi(\mathbf{x}, t)$, a function of 4-momentum like $\tilde{\varphi}(p^\mu)$, or whatever basis you choose to write things in just like in normal QM. Strictly, the fields are operator-valued functions as they may not commute (for example it may be the case that $[\varphi(x_\mu), \varphi(y_\nu)] \neq 0$).
- All the examples given here feature only one kind of field and are quadratic in that field. This arguably makes the corresponding theories *classical* as we'll see later in this section.

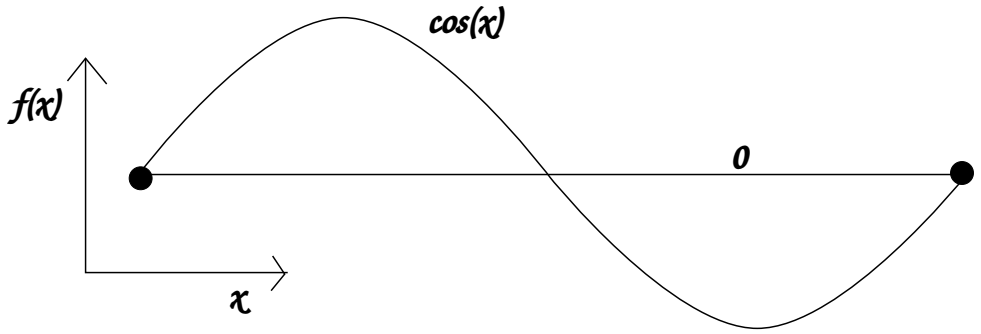

Figure 1: Two possible paths a particle could take between two points, assigned functions. The phase accumulated is a function of the function - *i.e.* a functional.

- Different types of field appear. I will stick to this convention: φ is a real scalar (spin-0) field, A_μ is a real vector (spin-1) field, ψ is a complex spinor (spin-1/2) field. There are many other types of field, these are just some common examples.
- The actions all include external currents j. The type of field dictates the type of current; for example the vector field A_μ requires a vector current with the same number of components, j^μ. Complex fields need complex currents, and so on.
- The action in each case involves an integral over a 'Lagrange density', which can be in real space or momentum space. For example the Klein-Gordon action is an integral over the Klein-Gordon Lagrange density:

$$\mathscr{L}_{\text{Klein-Gordon}} = \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 + j\varphi.$$

- Symmetries more or less constrain the theory. We will not have time to ruminate too long on this point here, but the modern way to construct field theories is to think of the symmetries a system obeys, then to write the most general Lagrange density compatible with those symmetries. In many cases, the resulting QFT makes very accurate testable predictions.

1.3 Functionals and Derivatives

A functional is very much like a function except that its domain is the set of functions rather than the set of (*e.g.*) real numbers. It is probably already clear why they are useful in QFT by considering standard QM: the famous explanation of the result of Young's two-slit experiment is that the photon takes every possible path between source and screen, and these paths interfere. A path through space is a function of space. For example, the straight-line path could be the trivial function $f(x) = 1$. Another path might be $f(x) = \cos(x)$, and so on. This is shown in Figure 1. We want to 'sum over histories' and weight each path by the quantum mechanical phase accumulated along it, multiplying with $\exp(iS[f])$.

In QFT we are considering fields $\varphi(\mathbf{x}, t)$ which are a function of both space and time, but otherwise everything is the same. The action is a functional of the possible field configurations, and we again weight each configuration by a phase $\exp(iS[\varphi])$. In both cases the 'sum' is actually a form of integral over possible fields, called a functional integral, but we'll come to that later on. We should learn to differentiate before we integrate, especially since only a very tiny number of functional integrals can even be carried out.

There is a simple generalization of normal derivatives to 'functional derivatives', considered shortly, but I've seen people come unstuck using it on more complicated functionals. For minimizations I find the easiest method is to explicitly vary the function. Say we consider paths with fixed endpoints like those in Figure 1, and that we'd like to know the shortest distance between the two points, with no additional weighting (no phases for example). The length L of a given path $f(x)$ is the sum of the lengths of the infinitesimal line segments:

$$\begin{aligned} L[f] &= \sum_{x_0}^{x_1} \sqrt{\mathrm{d}x^2 + \mathrm{d}f^2} \\ \mathrm{d}x \to 0 \quad &\downarrow \\ &= \int_{x_0}^{x_1} \mathrm{d}x \sqrt{1 + f'^2}. \end{aligned}$$

Now consider varying the path $f(x)$ to $f(x)+\epsilon(x)$ where the function $\epsilon(x)$ is small and completely arbitrary, except that it vanishes at the endpoints ($\epsilon(x_0)=\epsilon(x_1)=0$). In fact we'll consider a continuous parametrization between the two 'extremes': $f(x)+\lambda\epsilon(x)$, $\lambda\in[0,1]$. The function f which minimizes $L[f]$ is the one where, for any ϵ, a tiny shift from $\lambda=0$ leaves L unchanged. That is, if f minimizes L then

$$\left.\frac{\partial L[f+\lambda\epsilon]}{\partial\lambda}\right|_{\lambda=0}=0.$$

Note that the derivative is just a standard partial derivative, as λ is just a real number. If f were changed into some more complicated object such as a vector field A^μ, the variation would change to $A^\mu+\lambda\epsilon^\mu$ - but λ is always just a real number. We find in the present case that

$$\left.\frac{\partial L[f+\lambda\epsilon]}{\partial\lambda}\right|_{\lambda=0}=0=\int_{x_0}^{x_1}\mathrm{d}x\epsilon'\frac{1}{\sqrt{1+f'^2}}$$

or, integrating ϵ' by parts,

$$0=\int_{x_0}^{x_1}\mathrm{d}x\epsilon\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{1+f'^2}}\right).$$

This must be true for all ϵ, which means that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{1+f'^2}}\right)=0.$$

The derivatives are total derivatives and this can be solved by simple integration to yield (with the boundary conditions)

$$f(x)=(f(x_1)-f(x_0))\frac{x-x_0}{x_1-x_0}+f(x_0)$$

and we have shown that the shortest distance between two points, in Euclidean space, is a straight line.

The method can be formalized somewhat by introducing the functional derivative, given the symbol:

$$\frac{\delta L[f]}{\delta f}.$$

Aside from behaving basically like a normal derivative, but one acting on functions rather than variables, there are three requirements of the functional derivative:

1. $\left[\frac{\delta}{\delta f},\int\mathrm{d}x\right]=0$ (it commutes with integrals)
2. $\left[\frac{\delta}{\delta f},\frac{d}{dx}\right]=0$ (it commutes with normal derivatives)
3. $\frac{\delta f(x)}{\delta f(y)}=\delta(x-y)$.

Redoing the example from before (terms in parentheses indicate the rule used):

$$
\begin{aligned}
L[f] &= \int_{x_0}^{x_1} \mathrm{d}x \sqrt{1+f'^2} \\
&\downarrow \quad (1) \\
\frac{\delta L}{\delta f(y)} &= -\int_{x_0}^{x_1} \mathrm{d}x \frac{1}{\sqrt{1+f'^2}} \frac{\delta f'(x)}{\delta f(y)} \\
&\downarrow \quad (2) \\
\frac{\delta L}{\delta f(y)} &= -\int_{x_0}^{x_1} \mathrm{d}x \frac{1}{\sqrt{1+f'^2}} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\delta f(x)}{\delta f(y)} \\
&= \int_{x_0}^{x_1} \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{\sqrt{1+f'^2}} \right) \frac{\delta f(x)}{\delta f(y)} \\
&\downarrow \quad (3) \\
&= \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2}} \right)
\end{aligned}
$$

so as required we lose the integral. In many actual calculations, especially minimizations, it is often easier to continue to use the full form with λ.

1.4 Euler Lagrange Equations

The most important example of functional differentiation for fields will be taking derivatives of the action with respect to fields. Consider a general action

$$
S[\varphi] = \int \mathrm{d}^4 x \mathscr{L}(\varphi, \partial_\mu \varphi)
$$

and minimize as before:

$$
\begin{aligned}
S[\varphi + \lambda\epsilon] &= \int \mathrm{d}^4 x \mathscr{L}(\varphi + \lambda\epsilon, \partial_\mu \varphi + \lambda \partial_\mu \epsilon) \\
\left.\frac{\partial S[\varphi+\lambda\epsilon]}{\partial \lambda}\right|_{\lambda=0} &= 0 = \int \mathrm{d}^4 x \, \frac{\partial \mathscr{L}}{\partial(\varphi+\lambda\epsilon)} \frac{\partial(\varphi+\lambda\epsilon)}{\partial\lambda} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\varphi + \lambda\partial_\mu\epsilon)} \left.\frac{\partial(\partial_\mu\varphi+\lambda\partial_\mu\epsilon)}{\partial\lambda}\right|_{\lambda=0} \\
0 &= \int \mathrm{d}^4 x \left(\frac{\partial \mathscr{L}}{\partial\varphi}\epsilon + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\varphi)} \partial_\mu \epsilon \right) \\
&= \int \mathrm{d}^4 x \epsilon \left(\frac{\partial \mathscr{L}}{\partial\varphi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu\varphi)} \right)
\end{aligned}
$$

with an integration by parts in the final step. As before the arbitrariness of the field ϵ implies that for the integral to be zero the term in parentheses must be zero, and we have the Euler Lagrange equation for the field minimizing the action:

$$
\boxed{\frac{\partial \mathscr{L}}{\partial\varphi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu\varphi)} = 0.} \tag{2}
$$

By varying the actions in Equation 1, or equivalently sticking their Lagrange densities into Equation 2, we arrive at the Euler Lagrange equations for the respective fields. We find the equations which lent their names to the actions in each case:

$$
\begin{aligned}
\left(\Box + m^2\right)\varphi(x^\mu) &= j(x^\mu) \qquad \text{(Klein-Gordon)} \\
\partial^\nu F_{\mu\nu}(x^\nu) &= j_\mu(x^\nu) \qquad \text{(Maxwell)} \\
(i\partial_t - H)\psi(\mathbf{x}, t) &= j(\mathbf{x}, t) \qquad \text{(Schrödinger).}
\end{aligned}
$$

The j acts as a source for the respective fields (most obvious perhaps in the Maxwell case, where the source is an electromagnetic current). The standard equations are defined for $j \equiv 0$; the inhomogeneous Schrödinger equation, for instance, is a strange object, but as we will see later we could set $j(\mathbf{x}, t) = \delta^3(\mathbf{x})\delta(t)$ to find the Green's function, a very useful operator[2]. In the Schrödinger case the variation can be done with respect to either ψ or $\psi^\dagger$, but the resulting equations are merely hermitian conjugates of one another.

1.5 Propagators and Green's Functions

The method of Green's functions is a beautiful and powerful piece of mathematics. George Green lived in a windmill like Jonathan Creek - but whereas Creek solved hammy and implausible mysteries in the late 1990s, Green solved differential equations in the 1830s. Green realized that if you can find the response of a system to a δ-function impulse you know the response to any driving function, as you can make any function from an integral over δs. You can therefore write down a general solution to the equation. Green's functions generally have a physical significance in themselves, and we will see that in QFT they give the propagators for moving particles around.

Take as a simple example the Klein-Gordon equation. Rather than a general external current we would like to solve the equation for a δ-function kick (assuming the field lives in $3+1$D):

$$\left(\Box + m^2\right) G(x) = \delta^4(x).$$

This is easily solved by Fourier transform, defining

$$G(x) \triangleq \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-ip \cdot x) \tilde{G}(p)$$

where $p \cdot x \triangleq p_\mu x^\mu$, giving

$$\left(\Box + m^2\right) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-ip \cdot x) \tilde{G}(p) = \delta^4(x).$$

We recall a particular representation of the Green's function, and pull the differential operator through the p integral which it doesn't act on:

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{G}(p) \left(\Box + m^2\right) \exp(-ip \cdot x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-ip \cdot x)$$

so

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{G}(p) \left(m^2 - p^2\right) \exp(-ip \cdot x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-ip \cdot x)$$

and finally we have the momentum-space Green's function

$$\tilde{G}(p) = \frac{-1}{p^2 - m^2}.$$

Fourier transforming back to real space gives the Green's function

$$G(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-1}{p^2 - m^2} \exp(-ip \cdot x).$$

The fact that the momentum-space function is easier to work with is a regular feature of QFTs, as we often care about translationally invariant systems. It is also often the case that our boundary conditions constrain p rather than x making this basis particularly convenient. It's often easier to work in p and Fourier transform at the end if necessary. If we'd now like to solve the equation for some arbitrary but specified current $j(x)$ we have the solution by adding δ-functions as follows:

[2] A similar method applied instead to the time independent Schrödinger equation leads to the Lippmann-Schwinger equation in scattering theory.

$$\begin{aligned}\left(\Box + m^2\right) G\left(x-y\right) &= \delta^4\left(x-y\right) \\ \int \mathrm{d}^4 y\, j\left(y\right) \rightarrow &\quad \downarrow \\ \left(\Box + m^2\right)\left[\int \mathrm{d}^4 y G\left(x-y\right) j\left(y\right)\right] &= j\left(x\right).\end{aligned}$$

That is, the Green's function 'propagates' the disturbance in the field caused by the input current $j\left(y\right)$ to the field solving the Klein-Gordon equation:

$$\begin{aligned}\varphi\left(x\right) &= \varphi_0\left(x\right) + \int \mathrm{d}^4 y G\left(x-y\right) j\left(y\right) \\ &= \varphi_0\left(x\right) + \int \mathrm{d}^4 y G\left(x-y\right)\left(\Box_y + m^2\right)\left[\int \mathrm{d}^4 z G\left(y-z\right) j\left(z\right)\right] \\ &= \varphi_0\left(x\right) + \int \mathrm{d}^4 y G\left(x-y\right)\left(\Box_y + m^2\right)\left[\int \mathrm{d}^4 z G\left(y-z\right)\left(\Box_z + m^2\right)\left[\int \mathrm{d}^4 w G\left(z-w\right) j\left(w\right)\right]\right] \\ &\quad \textit{etc. etc.}\end{aligned}$$

where $\varphi_0\left(x\right)$ was a solution before the current was introduced. The physical interpretation of the Green's function is that it is the propagator for the φ field[3]. Later on, when considering interacting theories, j may itself be influenced by the field configuration, *i.e.* $j = j\left[\varphi\right]$. In this case the propagator method outlined here becomes very useful in defining a perturbative series for solving the problem.

Note that if we had the action written in momentum space we could just have 'read off' the propagator as the inverse of the function appearing between the fields in the quadratic term:

$$\begin{aligned}S_{Klein\,Gordon}\left[\tilde{\varphi},\, \tilde{j} \equiv 0\right] &= \int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4}\left(\frac{1}{2}\tilde{\varphi}\left(p^2 - m^2\right)\tilde{\varphi}\right) \\ &= \int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4}\left(-\frac{1}{2}\tilde{\varphi}\tilde{G}^{-1}\tilde{\varphi}\right).\end{aligned}$$

This is generically true: the inverse propagator is sandwiched inside the quadratic term in the action. The reason becomes apparent when we consider functional integrals later on.

1.6 Gauge Fixing*

This section will not be covered in lectures and is for the interest of people already familiar with gauge theories. Reading off the propagator for the Maxwell field is a bit tricky as it requires re-arranging into the form

$$S_{Maxwell}\left[A_\mu\right] = \int \mathrm{d}^4 x \frac{1}{2} A_\mu \left(G^{-1}\right)^{\mu\nu} A_\nu.$$

Starting from the form above

$$S_{Maxwell}\left[A_\mu,\, j^\mu\right] = \int \mathrm{d}^4 x \left(-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + j^\mu A_\mu\right)$$

integration by parts leads to the form

$$S_{Maxwell}\left[A_\mu,\, j^\mu\right] = \int \mathrm{d}^4 x \left(\frac{1}{2} A^\nu\left(\Box A_\nu - \partial^\mu \partial_\nu A_\mu\right) + j^\mu A_\mu\right)$$

which can now be written in the desired form by the introduction of a metric:

$$S_{Maxwell}\left[A_\mu,\, j^\mu\right] = \int \mathrm{d}^4 x \left(\frac{1}{2} A^\nu\left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right) A^\mu + j^\mu A_\mu\right)$$

[3]Prof. Hannay points out that strictly the Green's function and propagator are temporal Fourier transforms of one another, with the Green's function depending on energy and the propagator on time. In the literature the terms are often used interchangeably.

and in momentum space

$$S_{Maxwell}\left[\tilde{A}_\mu,\,\tilde{j}^\mu\right]=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{1}{2}\tilde{A}^\nu\left(-\eta_{\mu\nu}p^2+p_\mu p_\nu\right)\tilde{A}^\mu+\tilde{j}^\mu\tilde{A}_\mu\right)$$

where $\tilde{A}(p)=\tilde{A}(-p)$ for simplicity. This suggests the propagator will come from inverting the Euler Lagrange equation

$$\left(-\eta_{\mu\nu}p^2+p_\mu p_\nu\right)G^{\nu\rho}=\delta^\rho_\mu.$$

The problem with this is that the term in parentheses is non-invertible. It clearly has at least one zero eigenvalue in the form of p^μ:

$$\left(-\eta_{\mu\nu}p^2+p_\mu p_\nu\right)p^\mu=0$$

and since the determinant of an operator is the product of its eigenvalues the determinant must be zero, and the operator non-invertible.

Our gauge field A_μ has four degrees of freedom ($\mu\in[0,3]$), whereas real photons only have two degrees of freedom (the two possible polarizations). The additional redundancy is known as a gauge freedom, and needs to be restricted. This can be done by modifying the action, for example by adding

$$S_{Gauge\,fix}\left[A_\mu\right]=\int\mathrm{d}^4x\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2.$$

When $\xi\to0$ the Lorenz gauge $\partial_\mu A^\mu=0$ is rigidly enforced. The effect of this additional term on the Euler Lagrange equation is

$$\left(-\eta_{\mu\nu}p^2+\left(1-\frac{1}{\xi}\right)p_\mu p_\nu\right)\tilde{G}^{\nu\rho}=\delta^\rho_\mu$$

which is now invertible for all $\xi<\infty$. The solution is (it's a 'stick in and check' job rather than a 'calculate'):

$$\tilde{G}^{\mu\nu}=\frac{-1}{p^2}\left(\eta^{\mu\nu}+(\xi-1)\frac{p^\mu p^\nu}{p^2}\right).$$

Picking a ξ now fixes a gauge: $\xi=0$ is the Lorenz gauge, $\xi=1$ the Feynman gauge, $\xi=3$ the Yennie gauge (useful for a specific problem). Physical results do not depend on the choice of gauge, which merely reflects a mathematical redundancy built into the equations. Gauge fixing is a vital procedure in gauge theories, as this section has hopefully demonstrated.

1.7 Interaction Terms

The field theories considered so far have all been quadratic in the fields, with no theory featuring multiple field types. We can easily ('easily'!) find the propagators for such theories, and then we can find the response to any external current we choose to apply. There is no back-reaction on the current from the field.

QFT comes into its own when we consider *interactions*. These could be between two fields, such as in quantum electrodynamics (QED):

$$S_{\mathrm{QED}}\left[A_\mu,\psi\right]=\int\mathrm{d}^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\overline{\psi}\left(\gamma^\mu\left(p_\mu-eA_\mu\right)-m\right)\psi\right]$$

or of a field with itself, as in φ^4 theory:

$$S_{\varphi^4}\left[\varphi\right]=\int\mathrm{d}^4x\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}m^2\varphi^2+\frac{\lambda}{4!}\varphi^4\right]$$

or the whimsically-named sine-Gordon theory:

$$S_{\text{sine-Gordon}}\left[\varphi\right]=\int\mathrm{d}^4x\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}m^2\varphi^2+\cos\left(\varphi\right)\right].$$

There is a good reason to claim that all the non-interacting (*i.e.* Gaussian) theories, including Maxwell's theory of light, are *classical*. This is heavily debated, and is a point we will return to in later chapters after we have more tools at our disposal.

2 Partition Functions

The working of this section again sticks very closely to the notes of the Oxford Theoretical Physics course by John Chalker and André Lukas, which I have linked to on my website [1].

2.1 Gaussian Integrals

Consider the classic Gaussian integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}v \exp\left(-\frac{1}{2}av^2\right)$$

which is solved by squaring to give a 2D Gaussian integral, changing variables to plane polar coördinates, and square rooting:

$$\begin{aligned} I^2 &= \iint_{-\infty}^{\infty} \mathrm{d}v_1 \mathrm{d}v_2 \exp\left(-\frac{1}{2}a\left(v_1^2+v_2^2\right)\right) \\ I^2 &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}r r \exp\left(-\frac{1}{2}ar^2\right) \\ I^2 &= \frac{2\pi}{a} \\ I &= \sqrt{\frac{2\pi}{a}}. \end{aligned}$$

We could have written the first line as

$$I^2 = \iint_{-\infty}^{\infty} \mathrm{d}^2|v\rangle \exp\left(-\frac{1}{2}\langle v|A|v\rangle\right)$$

for 2D vector $|v\rangle \triangleq (v_1, v_2)^T$ and 2×2 matrix $A_2 \triangleq \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. In this case the result is perhaps more naturally written

$$I_2 = (2\pi)^{1/2} \left(\det\left(A_2\right)\right)^{-1/2}$$

where the subscript indicates a 2D space. Generalizing, we see that

$$I_n = (2\pi)^{n/2} \left(\det\left(A_n\right)\right)^{-1/2}$$

(the factor of n is taken care of naturally in the second term by the determinant). In effect we are solving n-dimensional Gaussian integrals. We can also generalize to arbitrary non-diagonal matrices provided all the eigenvalues are positive definite, since a unitary transformation will diagonalize again:

$$\begin{aligned} J &= \int \mathrm{d}^n|v\rangle \exp\left(-\frac{1}{2}\langle v|A_n|v\rangle\right) \\ &\downarrow \quad A_n = U^\dagger D_n U \\ J &= \int \mathrm{d}^n|v\rangle \exp\left(-\frac{1}{2}\left(\langle v|U^\dagger\right) D_n \left(U|v\rangle\right)\right) \\ &\downarrow \quad |w\rangle = U|v\rangle \\ J &= \int \mathrm{d}^n|w\rangle \left|\det\left(U\right)\right| \exp\left(-\frac{1}{2}\langle w|D_n|w\rangle\right) \\ &= (2\pi)^{n/2} \left(\det\left(D_n\right)\right)^{-1/2} \\ &\equiv (2\pi)^{n/2} \left(\det\left(A_n\right)\right)^{-1/2} \end{aligned}$$

where the modulus of the Jacobian for the variable change is given by $\left|\det\left(U\right)\right| \equiv 1$, and the determinant of the diagonal matrix $D_n \triangleq \mathrm{diag}\left(a_1, a_2, \ldots, a_n\right)$ and the original matrix A_n are identical since U is unitary.

We can additionally introduce a complete set of orthonormal basis states via a resolution of the identity:

$$\mathbb{I} \equiv \sum_{i=1}^{n} |e_i\rangle\langle e_i|$$

giving

$$(2\pi)^{n/2} \left(\det\left(A_n\right)\right)^{-1/2} = \int \mathrm{d}^n|v\rangle \exp\left(\sum_{i,j}^{n} -\frac{1}{2}\langle v|e_i\rangle\langle e_i|A_n|e_j\rangle\langle e_j|v\rangle\right).$$

In fact there is nothing to stop us taking the dimension of the vectors $n \to \infty$; the matrix $A_{n\to\infty}$ then becomes an operator. The infinite dimensional vectors become fields $|v\rangle \to \varphi$, the infinite sums of basis vectors become projections into a continuous basis (such as position space or momentum space; I'll pick position) $\sum_{i=1}^{n} |e_i\rangle\langle e_i| \to \int \mathrm{d}x|x\rangle\langle x|$, and the infinite product of integrals is known as a functional integral. The combination is written

$$\mathscr{Z} = \int \mathscr{D}\varphi \exp\left(\int \mathrm{d}x \int \mathrm{d}y \left(-\frac{1}{2}\varphi\left(x\right) A\left(x,y\right) \varphi\left(y\right)\right)\right).$$

The symbol $\mathscr{Z}$ suggests a partition function from statistical mechanics, and indeed this is the name we give the object[4]:

$$\mathscr{Z} = \int \mathscr{D}\varphi \exp\left(iS\left[\varphi\right]\right) \qquad \text{partition function.}$$

There is a deep link between statistical mechanics and quantum field theory which we will not have time to go into. In much the same manner that switching $t \to it$ in the Schrödinger equation produces the heat equation, switching $t \to it$ in QFT produces statistical mechanics. The process is known as Wick rotation - a $\pi/2$ rotation in the complex time plane invented by Gian-Carlo Wick.

2.2 The First Trick in the Book

Let's re-introduce the current into the partition function and work in momentum space:

$$\mathscr{Z}\left[\tilde{j}\right] = \int \mathscr{D}\tilde{\varphi} \exp\left(i \int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4} \left(-\frac{1}{2}\tilde{\varphi}\tilde{G}^{-1}\tilde{\varphi} + \tilde{j}\tilde{\varphi}\right)\right)$$

and for definiteness we'll take the Green's function to be the Klein-Gordon propagator

$$\tilde{G}\left(p\right) = \frac{1}{m^2 - p^2}.$$

The current term is linear in the field, so can be eliminated by completing the square (combining it into the quadratic term). The neatest way to do this, and the one which generalizes most straightforwardly to more complicated fields, is to change field variables:

$$\tilde{\varphi}\left(p\right) = \tilde{\Phi}\left(p\right) + \tilde{\phi}\left(p\right)$$

where $\tilde{\Phi}$ is some fixed configuration, and $\tilde{\phi}$ is a variation about this configuration, so that

$$\mathscr{D}\tilde{\varphi} = \mathscr{D}\tilde{\phi}.$$

The result is

$$\mathscr{Z}\left[\tilde{j}\right] = \int \mathscr{D}\tilde{\phi} \exp\left(i \int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4} \left(-\frac{1}{2}\tilde{\Phi}\tilde{G}^{-1}\tilde{\Phi} - \frac{1}{2}\tilde{\phi}\tilde{G}^{-1}\tilde{\phi} - \frac{1}{2}\tilde{\Phi}\tilde{G}^{-1}\tilde{\phi} - \frac{1}{2}\tilde{\phi}\tilde{G}^{-1}\tilde{\Phi} + \tilde{j}\left(\tilde{\Phi} + \tilde{\phi}\right)\right)\right).$$

In the case of the Klein Gordon field the inverse propagator $G^{-1} = \Box + m^2$ has two spacetime derivatives in the first term meaning that if we do two by-parts integrals we can show

[4] Eagle-eyed observers may note that an i has been introduced. The Gaussian integrals converge for complex arguments provided the real part of the exponent is negative.

$$\int \mathrm{d}^4x \left(-\frac{1}{2}\phi G^{-1}\Phi\right) = \int \mathrm{d}^4x \left(-\frac{1}{2}\Phi G^{-1}\phi\right).$$

In fact this should be true in most cases: the quadratic term in the action almost always comes from the kinetic energy term in the Lagrangian, and this term is a function of the derivative of the field rather than the field itself, so is actually quadratic in the *derivative* of the field. Continuing, then,

$$\mathscr{Z}\left[\tilde{j}\right] = \exp\left(i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(-\frac{1}{2}\tilde{\Phi}\tilde{G}^{-1}\tilde{\Phi}+\tilde{j}\tilde{\Phi}\right)\right)\int \mathscr{D}\tilde{\phi}\exp\left(i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(-\frac{1}{2}\tilde{\phi}\tilde{G}^{-1}\tilde{\phi}-\tilde{\Phi}\tilde{G}^{-1}\tilde{\phi}+\tilde{j}\tilde{\phi}\right)\right)$$

and if we choose the fixed, 'mean' field to be

$$\tilde{\Phi} = \tilde{j}\tilde{G}$$

the final two terms cancel, giving the result

$$\boxed{\mathscr{Z}\left[\tilde{j}\right] = \exp\left(i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{2}\tilde{j}\tilde{G}\tilde{j}\right)\mathscr{Z}_0} \tag{3}$$

where $\mathscr{Z}_0 \triangleq \mathscr{Z}\left[j \equiv 0\right]$ is the standard Gaussian, which we can do. In fact, the infinite number which comes from $\mathscr{Z}_0$ is of little interest, and we treat $\mathscr{Z}_0$ as a normalization. This infinite number derives from the infinite 'zero point energy' of the free field. It's one of the many infinities in QFT, but in this case a reasonable one: the total energy of an infinitely large field is going to be infinity, but it's only energy *differences* we're interested in.

Note that the first trick in the book really just amounts to completing the square.

2.3 Functional Averages

We define the functional average of a functional $\mathcal{O}\left[\varphi\right]$ to be

$$\boxed{\left\langle\mathcal{O}\right\rangle \triangleq \frac{\int \mathscr{D}\varphi\mathcal{O}\left[\varphi\right]\exp\left(iS\left[\varphi\right]\right)}{\int \mathscr{D}\varphi\exp\left(iS\left[\varphi\right]\right)} = \mathscr{Z}_0^{-1}\int \mathscr{D}\varphi\mathcal{O}\left[\varphi\right]\exp\left(iS\left[\varphi\right]\right).} \tag{4}$$

In the case of QFT, with S some action, such a functional average weighted by $\exp\left(iS\right)$ is known as the 'vacuum expectation value' (VEV). You will sometimes see it written

$$\left\langle\mathcal{O}\right\rangle \equiv \left\langle 0|\mathcal{O}|0\right\rangle \equiv \left\langle\Omega|\mathcal{O}|\Omega\right\rangle.$$

In the latter two cases the ket $|0\rangle$ or $|\Omega\rangle$ corresponds to the vacuum state, suggesting the interpretation that the VEV is the amplitude for the operator $\mathcal{O}$ to take the vacuum state back to itself.

One functional average of particular interest is the '2-point correlator'

$$\left\langle\varphi\left(x_2\right)\varphi\left(x_1\right)\right\rangle = \mathscr{Z}_0^{-1}\int \mathscr{D}\varphi\varphi\left(x_2\right)\varphi\left(x_1\right)\exp\left(iS\left[\varphi\right]\right)$$

which represents the amplitude for a particle in the field φ to propagate from spacetime point x_1^μ to point x_2^μ. There is a neat trick to evaluating such objects, making use of Equation 3. From the definition $\mathscr{Z}\left[j\right]$

$$\mathscr{Z}\left[j\right] = \int \mathscr{D}\varphi\exp\left(-i\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4y\varphi\left(x\right)G^{-1}\left(x,y\right)\varphi\left(y\right)+i\int \mathrm{d}^4xj\left(x\right)\varphi\left(x\right)\right)$$

where the form of the inverse Green's function has been generalized slightly to potentially be a function of two spacetime points (this is the most general form of propagator). We can carry out a functional differentiation with respect to j *under the integral* (a classic Feynman trick):

$$-i\frac{\delta\mathscr{Z}\left[j\right]}{\delta j\left(x_2\right)} = \int \mathscr{D}\varphi\varphi\left(x_2\right)\exp\left(-i\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4y\varphi\left(x\right)G^{-1}\left(x,y\right)\varphi\left(y\right)+i\int \mathrm{d}^4xj\left(x\right)\varphi\left(x\right)\right)$$

and

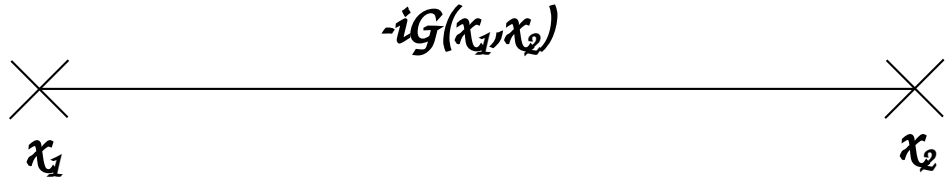

Figure 2: The 2-point correlator is equal to the propagator (up to a constant in this case through a choice of normalization). It gives the amplitude for a particle to propagate from spacetime point x_1 to point x_2.

$$\left(-i\frac{\delta}{\delta j\left(x_1\right)}\right)\left(-i\frac{\delta}{\delta j\left(x_2\right)}\right)\mathscr{Z}\left[j\right]=\int\mathscr{D}\varphi\varphi\left(x_2\right)\varphi\left(x_1\right)\exp\left(-i\frac{1}{2}\int\mathrm{d}^4x\int\mathrm{d}^4y\varphi\left(x\right)G^{-1}\left(x,y\right)\varphi\left(y\right)+i\int\mathrm{d}^4xj\left(x\right)\varphi\left(x\right)\right)$$

meaning the 2-point correlator is given by

$$\left(-i\frac{\delta}{\delta j\left(x_1\right)}\right)\left(-i\frac{\delta}{\delta j\left(x_2\right)}\right)\frac{\mathscr{Z}\left[j\right]}{\mathscr{Z}_0}\bigg|_{j\equiv0}=\left\langle\varphi\left(x_2\right)\varphi\left(x_1\right)\right\rangle.$$

From Equation 3 we have that

$$\frac{\mathscr{Z}\left[j\right]}{\mathscr{Z}_0}=\exp\left(i\int\mathrm{d}^4x\int\mathrm{d}^4y\frac{1}{2}j\left(x\right)G\left(x,y\right)j\left(y\right)\right)$$

and the 2-point correlator evaluates to

$$\begin{aligned}\left\langle\varphi\left(x_2\right)\varphi\left(x_1\right)\right\rangle &= \left(-i\frac{\delta}{\delta j\left(x_1\right)}\right)\left(-i\frac{\delta}{\delta j\left(x_2\right)}\right)\exp\left(i\int\mathrm{d}^4x\int\mathrm{d}^4y\frac{1}{2}j\left(x\right)G\left(x,y\right)j\left(y\right)\right)\bigg|_{j\equiv0}\\ &= \left(-i\frac{\delta}{\delta j\left(x_1\right)}\right)\int\mathrm{d}^4yG\left(x_2,y\right)j\left(y\right)\exp\left(i\int\mathrm{d}^4x\int\mathrm{d}^4y\frac{1}{2}j\left(x\right)G\left(x,y\right)j\left(y\right)\right)\bigg|_{j\equiv0}\\ &= -iG\left(x_1,x_2\right)\end{aligned}$$

assuming that $G\left(x,y\right)\equiv G\left(y,x\right)$ after the first line[5]. Thus we see that the 2-point correlator *is* the propagator for the field, which makes sense given that it propagates a particle excitation between 2 points. This is shown diagrammatically in Figure 2.

As an interesting aside, note the mathematical similarity of the j tricks to a Fourier transform:

$$\begin{aligned}\mathscr{Z}_0^{-1}\int\mathscr{D}\tilde{\varphi}\left\{\exp\left(i\int\frac{\mathrm{d}^4p}{\left(2\pi\right)^4}\left(-\frac{1}{2}\right)\tilde{\varphi}\tilde{G}^{-1}\tilde{\varphi}\right)\right\}\exp\left(i\int\frac{\mathrm{d}^4p}{\left(2\pi\right)^4}\tilde{j}\tilde{\varphi}\right) &= \left\{\exp\left(i\int\frac{\mathrm{d}^4p}{\left(2\pi\right)^4}\left(+\frac{1}{2}\right)\tilde{j}\tilde{G}\tilde{j}\right)\right\}\\ cf.\qquad\int\mathrm{d}x\left\{\exp\left(is\left(x\right)\right)\right\}\exp\left(ipx\right) &= \left\{\exp\left(i\tilde{s}\left(p\right)\right)\right\}\end{aligned}$$

(curly brackets introduced to emphasise the quantity being transformed) and derivatives with respect to j bring down a φ:

$$\begin{aligned}-i\frac{\delta}{\delta j\left(x\right)} &\rightarrow \varphi\left(x\right)\\ cf.\qquad -i\frac{\partial}{\partial p} &\rightarrow x.\end{aligned}$$

I've never seen much discussion of this, but I believe there is some use in loop quantum gravity.

[5] If we don't assume this we get $-\frac{i}{2}\left(G\left(x_1,x_2\right)+G\left(x_2,x_1\right)\right)$ instead which is a bit clumsier to carry around.

2.4 Interaction Terms Again

Note that by the reasoning in the last section the functional average of an arbitrary functional $\mathcal{O}[\varphi]$ can be evaluated by

$$\langle \mathcal{O} \rangle \triangleq \frac{\int \mathscr{D}\varphi \mathcal{O}[\varphi] \exp(iS[\varphi])}{\int \mathscr{D}\varphi \exp(iS[\varphi])} = \mathcal{O}\left[-i\frac{\delta}{\delta j}\right] \frac{\mathscr{Z}[j]}{\mathscr{Z}_0}\bigg|_{j\equiv 0}$$

since the functional derivative pulls down a φ each time it acts. The method is even more powerful than it may appear, and it explains one of the mathematical paradoxes of QFT: given that pretty much the only functional integrals which are mathematically well-defined are Gaussians[6], how can we construct interesting (*i.e.* interacting) QFTs? The answer is that we only ever *need* to do Gaussian integrals! We then employ the mean field trick and functionally differentiate under the functional integral to generate whatever functions we like.

To see how this works let's take another look at interaction terms. We'll take as an example φ^4 theory:

$$\mathscr{W}[j] \triangleq \frac{\int \mathscr{D}\varphi \exp(iS_0[\varphi] + iS_{int}[\varphi] + iS_j[\varphi])}{\int \mathscr{D}\varphi \exp(iS_0[\varphi])} = \mathscr{Z}_0^{-1} \int \mathscr{D}\varphi \exp\left(i\int \mathrm{d}^4x \left(-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 + j\varphi\right)\right)$$

where the normalization $\mathscr{Z}_0^{-1}$ is now included as standard. We could reinterpret this expression as being the functional average of the interaction term:

$$\mathscr{W}_0 = \left\langle \exp\left(i\int \mathrm{d}^4x \frac{\lambda}{4!}\varphi^4\right)\right\rangle$$

(where $\mathscr{W}_0 \triangleq \mathscr{W}[j \equiv 0]$) in which case we can evaluate it as before using

$$\mathscr{W}_0 = \exp\left(i\int \mathrm{d}^4x \frac{\lambda}{4!}\left(-i\frac{\delta}{\delta j}\right)^4\right) \frac{\mathscr{Z}[j]}{\mathscr{Z}_0}\bigg|_{j\equiv 0}.$$

The answer hints at how we deal with interactions, since it suggests a series expansion in powers of λ involving propagators G. In general we have that

$$\mathscr{W}_0 = \exp\left(iS_{int}\left[-i\frac{\delta}{\delta j}\right]\right) \frac{\mathscr{Z}[j]}{\mathscr{Z}_0}\bigg|_{j\equiv 0}.$$

The object $\mathscr{W}[j]$ is known as a 'generating functional'; $j(x)$ *may* have a physical interpretation, but its primary use is to allow us to generate quantities of interest by taking functional derivatives with respect to it. This is a useful trick throughout all of physics.

[6] Prof. Berry has worked out a set of other (generally quite strange) cases where the functional integral can be carried out exactly using the saddle-point approximation.

3 n-point Functions

The best reference for this section is again reference [1].

3.1 Wick's Theorem

To streamline the notation slightly let's define

$$\begin{aligned}
\varphi(x_1) &\triangleq \varphi_1 \\
j(x_1) &\triangleq j_1 \\
\frac{\delta}{\delta j_1} &\triangleq \delta_{j_1} \\
G(x_1, x_2) &\triangleq G_{12}
\end{aligned}$$

(the algebra gets a bit tedious otherwise). We already calculated the 2-point correlator and found that the result was the propagator for the field. We found that

$$\begin{aligned}
\langle \varphi_2 \varphi_1 \rangle &= (-i\delta_{j_1})(-i\delta_{j_2}) \exp\left(i \int \mathrm{d}^4x \int \mathrm{d}^4y \frac{1}{2} j_x G_{xy} j_y \right)\Big|_{j\equiv 0} \\
&= -iG_{12}.
\end{aligned}$$

What if we calculated the 3-point correlator? A quick check shows the answer to be zero, and the same for any n-point function for odd n. The next non-zero value is the 4-point function:

$$\begin{aligned}
\langle \varphi_1\varphi_2\varphi_3\varphi_4 \rangle &= (-i\delta_{j_4})(-i\delta_{j_3})(-i\delta_{j_2})(-i\delta_{j_1}) \exp\left(i \int \mathrm{d}^4x \int \mathrm{d}^4y \frac{1}{2} j_x G_{xy} j_y \right)\Big|_{j\equiv 0} \\
&= \left(-i\frac{\delta}{\delta j_4}\right)\left(-i\frac{\delta}{\delta j_3}\right)\left(-i\frac{\delta}{\delta j_2}\right)\left(\int \mathrm{d}^4x G_{x1} j_x\right) \exp\left(i \int \mathrm{d}^4x \int \mathrm{d}^4y \frac{1}{2} j_x G_{xy} j_y \right)\Big|_{j\equiv 0} \\
&= \left(-i\frac{\delta}{\delta j_4}\right)\left(-i\frac{\delta}{\delta j_3}\right)\left[-iG_{12} - i\int \mathrm{d}^4x G_{x1} j_x \int \mathrm{d}^4y G_{y2} j_y\right] \exp\left(i \int \mathrm{d}^4x \int \mathrm{d}^4y \frac{1}{2} j_x G_{xy} j_y \right)\Big|_{j\equiv 0} \\
&\vdots \\
&= -G_{12}G_{34} - G_{23}G_{14} - G_{24}G_{13}
\end{aligned}$$

(if you follow the algebra you'll agree with the streamlined notation!). Up to a constant prefactor, then, the 4-point function is the sum of possible contractions into 2-point correlators. In fact the result generalizes to all nonzero correlators. It is known as Wick's theorem:

$$\langle \varphi_1 \varphi_2 \dots \varphi_{2n} \rangle = \sum_{\substack{\text{possible} \\ \text{pairwise} \\ \text{contractions} \\ i=1,\, j=1,\dots}}^{i=2n,\, j=2n,\dots} G_{ij} G_{kl} \dots G_{yz}$$

3.2 The φ^4 vacuum

We now have everything in place to introduce interactions formally. Consider again the object

$$\mathscr{W}_0 = \left\langle \exp\left(i \int \mathrm{d}^4x \frac{\lambda}{4!} \varphi^4 \right) \right\rangle$$

where the exponential is defined through its Taylor series (and therefore the coupling λ is assumed small). To second order in the coupling we have

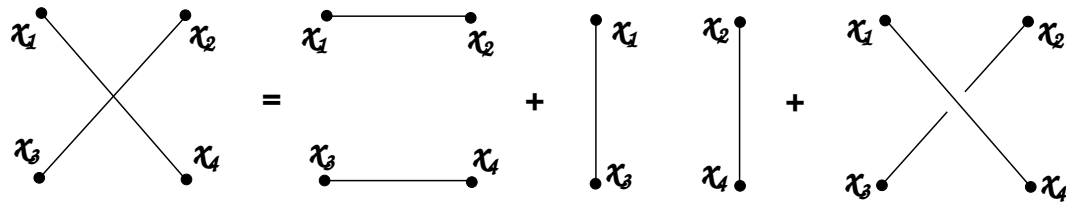

Figure 3: The 4-point function decomposes into all possible 2-point functions. This is an instance of the more general rule that n-point functions (*i.e.* any correlator one might care to imagine) decompose into products of 2-point functions. This is known as Wick's theorem.

$$\begin{aligned}\mathscr{W}_0 &= \left\langle 1 + i\frac{\lambda}{4!}\int \mathrm{d}^4x\varphi_x^4 + \left(i\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \varphi_x^4\varphi_y^4 + \ldots \right\rangle \\ &= 1 + i\frac{\lambda}{4!}\int \mathrm{d}^4x \left\langle\varphi_x^4\right\rangle + \left(i\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \left\langle\varphi_x^4\varphi_y^4\right\rangle + \ldots\end{aligned}$$

which can be evaluated using Wick's theorem (and the results from before) to give

$$\begin{aligned}\mathscr{W}_0 &= 1 + i\frac{\lambda}{4!}\int \mathrm{d}^4x \left(-3G_{xx}^2\right) + \frac{1}{2}\left(i\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \left(3^2G_{xx}^2G_{yy}^2 + 4^2 3^2 G_{xx}G_{yy}G_{xy}^2 + 4!G_{xy}^4\right) \\ &= 1 - i\frac{\lambda}{8}\int \mathrm{d}^4x G_{xx}^2 - \lambda^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\frac{1}{128}G_{xx}^2G_{yy}^2 + \frac{1}{16}G_{xx}G_{yy}G_{xy}^2 + \frac{1}{48}G_{xy}^4\right).\end{aligned}$$

Let's look at the combinatorics regarding the coefficients in the λ^2 term, as this is the really fiddly bit and probably whence QFT gets its reputation for difficulty. We want all possible contractions of

$$\varphi_x\varphi_x\varphi_x\varphi_x\varphi_y\varphi_y\varphi_y\varphi_y$$

these split into three types:

$$\begin{array}{cccc} \underbrace{\varphi_x\varphi_x} & \underbrace{\varphi_x\varphi_x} & \underbrace{\varphi_y\varphi_y} & \underbrace{\varphi_y\varphi_y} \\ \underbrace{\varphi_x\varphi_x} & \underbrace{\varphi_y\varphi_y} & \underbrace{\varphi_x\varphi_y} & \underbrace{\varphi_x\varphi_y} \\ \underbrace{\varphi_x\varphi_y} & \underbrace{\varphi_x\varphi_y} & \underbrace{\varphi_x\varphi_y} & \underbrace{\varphi_x\varphi_y} \end{array}$$

in the first case, the φ_x and φ_y combine separately; the first φ_x has 3 others to choose from, then the other pair is completely constrained. The total number of ways of forming this contraction, including the φ_y terms, is therefore 3^2, and combined with the prefactor this gives $\frac{3^2}{2\times(4!)^2} = \frac{1}{128}$. In the second case we want to form one $\varphi_x\varphi_x$ pair; there are 6 ways to do this, and the same for the $\varphi_y\varphi_y$. The remainder is $\varphi_x\varphi_x\varphi_y\varphi_y$ which is to be formed into $\varphi_x\varphi_y$ pairs, and there are 2 ways to do this in total. So the total prefactor is $\frac{2\times6^2}{2\times(4!)^2} = \frac{1}{16}$. Finally we want to form all possible $\varphi_x\varphi_y$ contractions. Let's allow the φ_x to choose partners from amongst the φ_y, just as the girls were allowed to choose from the boys in a ceilidh I once attended. The first φ_x has 4 φ_y to choose from, then the second φ_x has 3 φ_y, the third φ_x has 2 φ_y to choose from, and the last φ_x gets what she's given (in the ceilidh analogy she would be partnered with myself). This gives 4! combinations, so the total prefactor this time is $\frac{4!}{2\times(4!)^2} = \frac{1}{48}$.

The philosophy of what is happening here is very interesting. We are either finding the expectation value of the φ^4 interaction term with respect to the vacuum of the non-interacting theory, or equivalently we are finding the expectation value of the vacuum with respect to the interacting φ^4 theory. In the latter interpretation we see that interacting theories have infinitely rich vacua: rather than being empty, the vacuum contains particles coming into and out of existence in all possible ways which involve 4-vertices. Note that it's 4-vertices because we have a φ^4 interaction. If we had instead the QED interaction term $\sim A_\mu\psi^\dagger\psi$ the vertices would need one electron going in, $\psi^\dagger$, one coming out, ψ, and one photon A_μ either coming in or going out. The interaction term specifies the only types of vertex in the theory.

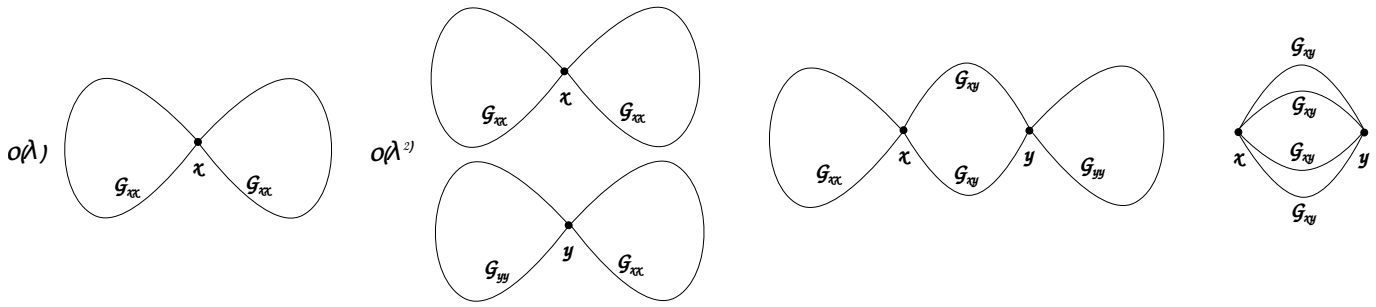

Figure 4: The φ^4 vacuum to second order in the coupling. The vacuum consists of all possible ways to combine the 4-vertex generated by the φ^4 interaction.

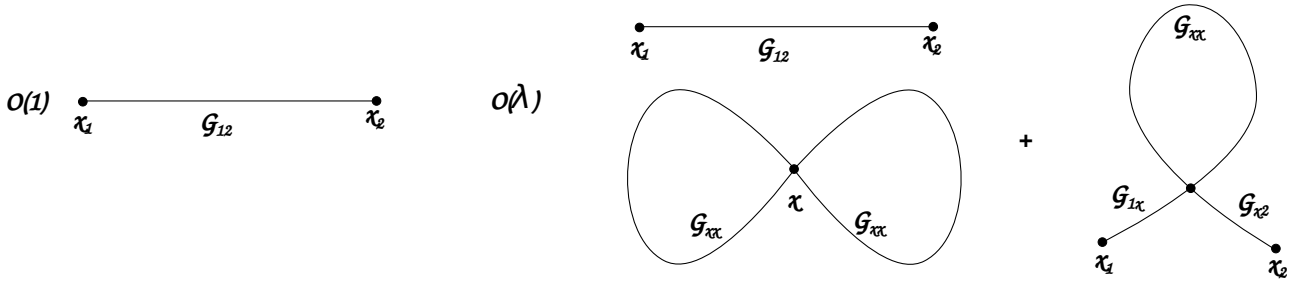

Figure 5: The 2-point correlator for φ^4 theory, to first order in the coupling. The first of the $\mathcal{O}(\lambda)$ diagrams contains a bubble.

Another way to think of the result is that the functional integral $\int \mathscr{D}\varphi$ varies the vacuum over all possible field configurations, with each configuration weighted by $\exp(iS[\varphi])$. This is quite natural when thinking of QFT as an extension of the Feynman path integral method of QM: in that case we consider 1D paths $f(x)$ between points in space and use the functional integral $\int \mathscr{D}f$ to sum over all possible paths, with each path again weighted by $\exp(iS[f])$. For this reason people sometimes refer to QM as a $0+1$D QFT: it's a QFT with a 'field' x which is a function only of time: $x(t)$.

3.3 The φ^4 n-point functions

In exactly the same way as we evaluated the vacuum in the last section we can evaluate n-point functions. The 2-point function in φ^4 theory, for example, is given by

$$\langle\varphi(x_2)\varphi(x_1)\rangle_\lambda = \frac{\int \mathscr{D}\varphi\varphi(x_2)\varphi(x_1)\exp(iS_0+iS_{int})}{\int \mathscr{D}\varphi\exp(iS_0)}$$

where λ suggests the interacting theory rather than the theory with $\lambda=0$, and to order λ^2 this expands to

$$\langle\varphi_2\varphi_1\rangle_\lambda = \langle\varphi_2\varphi_1\rangle_0 + i\frac{\lambda}{4!}\int \mathrm{d}^4x\,\langle\varphi_2\varphi_1\varphi_x^4\rangle_0 + \left(i\frac{\lambda}{4!}\right)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\,\langle\varphi_2\varphi_1\varphi_x^4\varphi_y^4\rangle_0 + \mathcal{O}(\lambda^3)$$

using the streamlined notation, and $\langle\ldots\rangle_0$ indicating the vacuum expectation value with respect to the non-interacting vacuum (the thing we calculated before) for clarity. Some work leads to the result (only first order in λ as second order already contains ten φ terms!)

$$\langle\varphi_2\varphi_1\rangle_\lambda = -iG_{12} + i\frac{\lambda}{4!}\int \mathrm{d}^4x\left(3iG_{12}G_{xx}^2 + 12iG_{1x}G_{xx}G_{x2}\right) + \mathcal{O}(\lambda^2).$$

The result is shown diagrammatically in Figure 5. The interaction term has added an infinite number of correction terms to the propagator, where terms with n vertices are suppressed by a factor $\left(\frac{\lambda}{4!}\right)^n$. These are known as 'quantum corrections'.

So can non-interacting (*i.e.* Gaussian) theories be quantum? It depends on how you define a quantum field theory as opposed to a classical one. One option might be to say that quantum theories are those where the probability for an event is given by summing amplitudes for different processes before taking a modulus square.

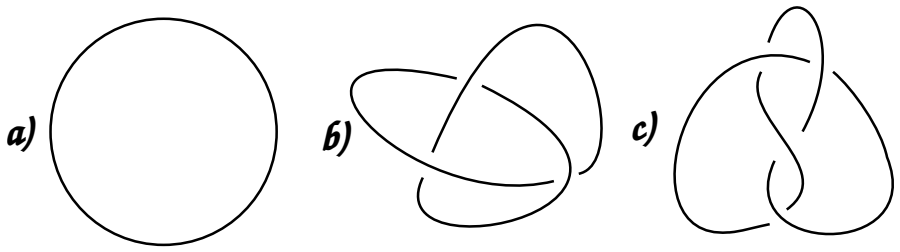

Figure 6: In non-interacting theories there are no vertices, so the only vacuum bubble possible is a closed loop (a). However, in the non-interacting Chern Simons theory the different embeddings of that loop count differently, and the different possible knots sum as independent amplitudes: (b) shows a trefoil knot, and (c) a figure-of-eight knot, whereas (a) is known as the *unknot*.

In Gaussian theories there can only be one diagram per process: there can be no quantum corrections to any n-point functions as there are no vertices[7]; similarly there is only one allowed vacuum bubble, a closed loop. So this definition doesn't distinguish quantum and classical for Gaussian models, although the lack of quantum corrections does rather suggest Gaussian theories are classical. Another option is to consider *second quantization* which we won't directly cover in this course. This is a rewriting of the fields as an infinite set of harmonic oscillator creation and annihilation operators, each sat at a different point in space (or, more commonly, momentum space). It's a rewriting of the field as a collection of particles, where we interpret the m^{th} excited state of the oscillator labelled by $\mathbf{p}$ as m particles of momentum $\mathbf{p}$. Both interacting and Gaussian theories can be second quantized, but classical fields cannot[8]. Probably the most agreed-upon definition of a quantum field theory is that it obeys *canonical commutation relations*, $[\varphi(x), \pi(y)] = i\delta(x-y)$, where π, the momentum conjugate to the field φ, is defined to be

$$\frac{\partial \mathscr{L}}{\partial \partial_0 \varphi(x^\mu)} = \pi(x^\mu).$$

This is ultimately the same as the second quantization argument. The question of whether Maxwell theory is quantum is still unresolved, though, because it is a very subtle matter to have it obey canonical quantization - involving gauge fixing - and is beyond the scope of Peskin and Schroeder, even!

An advanced note for the interested reader: there is a class of Gaussian theories which contains inequivalent vacuum bubbles. How can this happen, given that the only allowed diagram is a closed loop? The answer is that this class of theories, called *topological* QFTs, cares how the loop is embedded in the space. That is, it can tell what kind of knot the loop is tied in (see Figure 6). The different knots add as amplitudes, so this theory is undoubtedly quantum. Chern Simons theory is the classic example (super-advanced note: to be Gaussian the theory must be either Abelian, or non-Abelian in the Weyl gauge $A_a^0 = 0$).

3.4 Feynman Diagrams

Feynman diagrams have already been appearing throughout these notes as I assume most people have some familiarity with them already (and they're very intuitive).

Aside from giving a neat interpretation to the maths in terms of particle creation an annihilation, the real strength of Feynman diagrams comes when you can use them to jump straight to the answer rather than having to carry out the complicated combinatorics of the functional derivative method we've employed so far.

As noted in Section 3.1, Wick's theorem states that the vacuum expectation value of any combination of fields is always a product of propagators (the VEV decomposes into the sum of the VEVs of possible pairs of fields). Any VEV can therefore be expressed in terms of the propagator and vertices. First let's take the general case for a real scalar field:

$$S[\varphi] = \int \mathrm{d}^4x \int \mathrm{d}^4y \frac{1}{2}\varphi(x) G^{-1}(x,y) \varphi(y) + \int \mathrm{d}^4x \sum_{n>2} \alpha_n \frac{1}{n!}\lambda^n \varphi^n$$

[7] Astute readers will have noticed that the 4-point function for a non-interacting theory has 3 diagrams; we will see shortly that none of these contribute to physical processes.

[8] *Reductio ad absurdum* for the reader familiar with second quantization (say, pages 19-21 of Peskin and Schroeder): classical fields commute with their conjugate momenta, $[\varphi(x), \pi(y)] = 0$, implying their ladder operators commute, $\left[a_{\mathbf{k}}, a_{\mathbf{q}}^\dagger\right] = 0$. But then $\left[H, a_{\mathbf{k}}^\dagger\right] = 0$ and $a_{\mathbf{k}}^\dagger$ is not a ladder operator. QED.

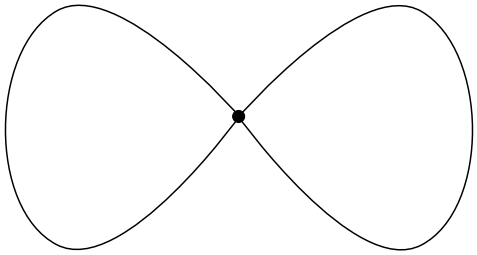

Figure 7: The figure-of-eight bubble diagram.

where the interaction term has been expressed as a Taylor series, which is always possible. The φ^4 case has $\alpha_4 = 1,\ \alpha_{\neq 4} = 0$. Then:

- for each fixed external field at spacetime point x_i^μ we assign a fixed vertex
- for each appearance of an internal n-vertex we assign $-i\alpha_n\lambda^n$
- for each instance of an internal line between points x^μ and y^μ we assign G_{xy} (if there are multiple fields in the theory each one will have a propagator; they may be trivial, *i.e.* just a number)
- we integrate over all internal spacetime variables
- for each diagram we multiply all of the above elements together and divide by the corresponding symmetry factor.

You might wonder where (i) the factor from the Taylor expansion of the operator $\mathcal{O}$ and (ii) the $\frac{1}{n!}$ from the interaction have gone. For (i), at order m the Taylor series for $\mathcal{O}$ will have a prefactor $1/m!$, but this will cancel with the $m!$ degeneracy due to interchanging the m vertices. For (ii), there will be n lines heading into each vertex, giving an $n!$ degeneracy, cancelling the $1/n!$ included in the definition above (for this reason). Both prefactors are therefore already accounted for.

For example, let's take the figure-of-eight bubble diagram appearing in Figures 4 and 5, reproduced in Figure 7. This has no external fields, one 4-vertex (since we were considering φ^4 theory this was the only allowed vertex), and two propagators going into (or out of) the same internal point (call it x^μ, but it's a dummy variable as we'll integrate over it since it's internal), so two factors of G_{xx} and an integral over d^4x. The symmetry factor is the number of diagrams equivalent to the one drawn. In this case either of the two lines could be reversed, or the two lines could be exchanged, and we wouldn't know - so the factor is $2 \times 2 \times 2$. Thus the diagram evaluates to

$$-\frac{\lambda}{8}\int \mathrm{d}^4x G_{xx}G_{xx}.$$

The method is generally a lot simpler in momentum space, where the propagator should be a simple function of one variable (assuming the problem has translational symmetry, which it does unless lattice effects are important[9]). There's the added advantage that in experiments we'll generally fix the ingoing and outgoing momenta rather than positions, so objects like

$$\langle \tilde{\varphi}\left(p_2\right)\tilde{\varphi}\left(p_1\right)\rangle$$

are a lot more natural than

$$\langle \varphi\left(x_2\right)\varphi\left(x_1\right)\rangle .$$

There's a major health warning accompanying the use of Feynman diagrams - you have to be extremely proficient to properly calculate the symmetry factors. It's far easier to make a mistake in the factors by this method than it is in the lengthy combinatoric method, or the even lengthier full method of taking δ_j functional derivatives. The method I use is to draw all the diagrams I can think of at the given order in the coupling, then to apply the combinatoric method, then try to see the symmetries in the diagrams afterwards. I think this is probably a standard way to go about things.

[9] Lattice effects will always prove important at some point - in condensed matter the spacing is the atomic spacing, and in particle physics it's around the Planck length. On these scales the field picture breaks down, and we need to deal with the effects carefully.

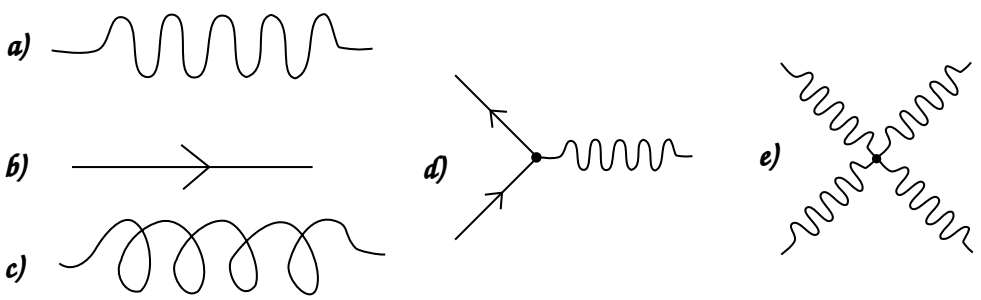

Figure 8: Common conventions for drawing Feynman diagrams: (a) bosonic propagator (*e.g.* photon γ, phonon φ, W or Z boson) , (b) fermionic propagator (*e.g.* electron $\psi^\dagger$ or $c^\dagger$, proton), (c) gluon propagator, (d) QED-type vertex (say, an electron and positron combine to a photon), (e) φ^4-type vertex (two φ particles scatter into two more, or four annihilate at a point). Being strict we should strike through the propagators in (d) and (e) because they're not themselves included in the vertices. In practice we often don't bother. Quite disturbingly, cutting off external legs in this manner is called 'amputation'.

The interpretation, due to Feynman and Stückelberg, of these diagrams as particles moving through spacetime is powerful but easily abused. To tell the story properly we need to introduce the propagators for more-interesting fields, and some more vertices, since real scalar fields are their own antiparticles. Complex fields allow for antiparticles as the complex conjugate of the field, *i.e.* if $\varphi^*(x)$ creates a particle at point x then $\varphi(x)$ creates an antiparticle, or equivalently annihilates a particle. The propagator for fermion fields (spin-$\frac{1}{2}$ or potentially $\frac{3}{2}$, $\frac{5}{2}$ *etc.*) is drawn as a line with an arrow on it, indicating the flow of fermion number. Since fermion number is conserved the divergence of arrows at a vertex is always zero (if one goes in, one must go out). Some common conventions are shown in Figure 8.

Bosons need not be conserved, so there are no arrows on bosonic propagators[10]. The lack of an arrow means in practice one has to add an arrow next to the line to indicate the direction of momentum transfer, otherwise sign errors may be introduced.

With all this in place we can interpret the diagrams as sequences of spacetime events. This is probably most easily done by considering some examples, which I do in Figure 9. An important point to note is that the addition to the diagrams of axes for space and time (in either orientation), which is commonly seen at undergraduate level, is totally misleading. There's no requirement that the spacetime events appearing in the diagrams be timelike separated, so the idea of a consistent time-ordering to the events is false. Internal spacetime points are integrated over anyway, so occur at all possible points in spacetime. The labels *may* be useful in certain limited cases, for example if we fix a number of external spacetime events in a given frame (probably the lab), but this is a rare case. In most practical cases the boundary conditions are 4-momenta anyway.

The interpretation of antiparticles as particles running backwards in time is acceptable but not necessary. The argument derives from complex conjugating the time-dependent Schrödinger equation:

$$\begin{aligned} i\partial_t\psi(\mathbf{x},t) &= \hat{H}\psi(\mathbf{x},t) \\ -i\partial_t\psi^*(\mathbf{x},t) &= \hat{H}\psi^*(\mathbf{x},t) \\ \therefore\ i\partial_t\psi^*(\mathbf{x},-t) &= \hat{H}\psi^*(\mathbf{x},-t) \end{aligned}$$

i.e. if ψ solves the Schrödinger equation with variable t, ψ^* is a solution with $-t$. Still, in relativistic theories the only really sensible quantities are spacetime events, so the idea of time 'flowing' in either direction is an additional, arguably incorrect, assertion. The quantum field might have a nonzero correlation $\langle\varphi(x^\mu)\varphi(y^\nu)\rangle$ between two spacetime events x^μ and y^ν, but the interpretation of this as a particle being created at one event and propagating to the other, or an antiparticle moving between the points in the opposite sense, is additional. Note that the 'time reversal $\equiv$ complex conjugation' argument doesn't hold for the Klein-Gordon equation

$$\left(\Box + m^2\right)\varphi = 0.$$

That equation governs bosonic scalar fields - even if the fields are complex, time reversal has no effect.

[10] This includes gluons, which get a special propagator symbol. I'm not entirely certain why this is, but one important feature of gluons is that their self-interactions (they have a 3-vertex and a 4-vertex) are extremely important because of quark confinement and 'asymptotic freedom'. It's possible the helical lines' vertices were considered easier to draw.

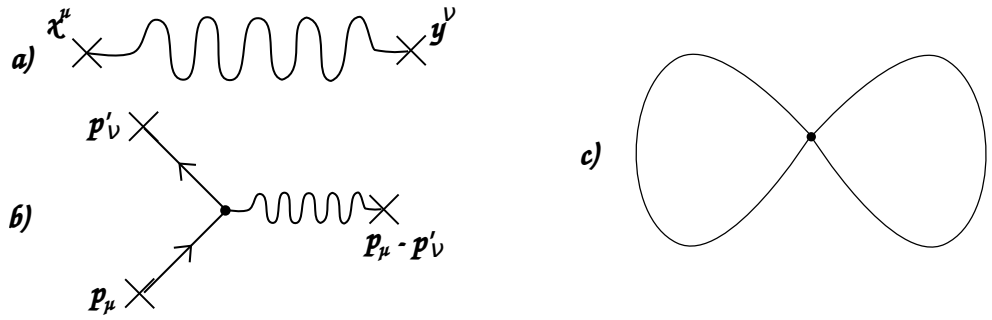

Figure 9: Three Feynman diagrams. In (a) a bosonic particle propagates from spacetime point x^μ to point y^ν. If we pick a frame of reference in which $x^0 < y^0$ the Feynman-Stückelberg interpretation would be to say the particle travels from $\mathbf{x}$ to $\mathbf{y}$, or alternatively the antiparticle travels backwards in time from $\mathbf{y}$ to $\mathbf{x}$. In (b) it is external momenta that are fixed rather than spacetime locations. An electron (say) with 4-momentum p_μ annihilates a positron with 4-momentum p'_ν, creating a photon (say) with 4-momentum $p_\mu - p'_\nu$. Note that vertices always conserve 4-momentum like this even when they're internal. An alternative interpretation could be that the electron emits the photon, and thus changes its 4-momentum. There is no need to interpret either fermion line as either particle or antiparticle. In (c) we have a vacuum bubble for a φ^4 interaction. We could say four particles are created at the vertex and annihilate in pairs; that two particle/antiparticle pairs are created then all annihilate; that a particle/antiparticle pair are created, separate then come together to annihilate, and as they do they hit into another particle/antiparticle pair doing the same but coming from the future into the past, *etc. etc.* . In case this isn't general enough to show that it's rather arbitrary how to interpret the diagrams, note that the vertex is internal so gets integrated over - this could be done either in real space or momentum space with the same result (dummy variable).

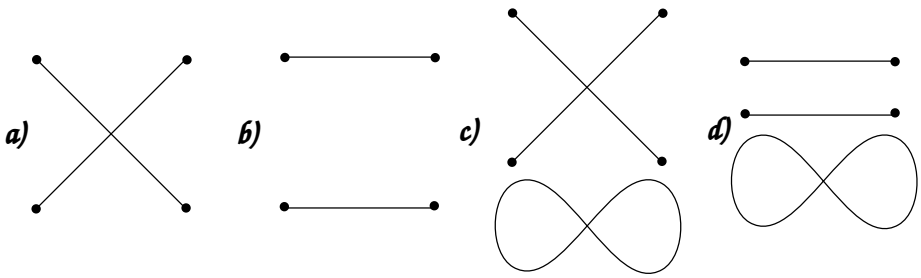

Figure 10: A connected diagram has all external legs connected. This does not rule out bubble diagrams: (a) a connected diagram; (b) a disconnected diagram; (c) a connected diagram with a bubble; (d) a disconnected diagram with a bubble. Unfortunately there's another usage of the word 'disconnected' which *does* mean diagrams with bubbles.

3.5 Cancelling Disconnected Terms

When calculating physically measurable quantities such as scattering amplitudes, it turns out that only diagrams where all external legs are connected together contribute. These diagrams are 'connected'. We could simply throw away disconnected terms, and often do in practice, but to be a little more mathematically rigorous we can instead consider a different generating functional which only generates connected diagrams. Note that the definition of 'connected' does not rule out bubble diagrams. Some clarification is given in Figure 10.

The relevant generating functional turns out to be simply

$$\mathscr{V}[j] \triangleq \ln \mathscr{W}[j]$$

where $\mathscr{W}$ is the full generating functional considered before. Note that this form causes:

$$\frac{\delta \mathscr{V}[j]}{\delta j} = \frac{1}{\mathscr{W}[j]} \frac{\delta \mathscr{W}[j]}{\delta j} = \mathscr{W}_j^{-1} \frac{\delta \mathscr{W}[j]}{\delta j}.$$

Now, recalling our streamlined notation:

$$\delta_1 \mathscr{Z}_j \quad = \quad \langle \varphi_1 \rangle_j$$

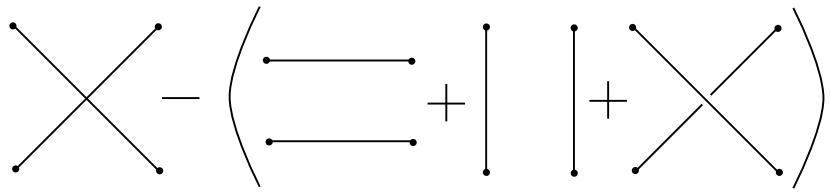

Figure 11: The connected 4-point function for φ^4 theory consists of the full 4-point function minus the disconnected diagrams. It is generated by the connected generating functional $\mathscr{V}_j$.

and for the 4-point function

$$\delta_4\delta_3\delta_2\delta_1\mathscr{Z}_j = \langle\varphi_1\varphi_2\varphi_3\varphi_4\rangle_j$$

and so on for higher n-point functions. Well actually we're going to use *even more streamlined notation* where these quantities now read

$$\begin{aligned}\delta_1\mathscr{Z}_j &= \langle\varphi_1\rangle_j &&\triangleq \overline{1}\\ \delta_4\delta_3\delta_2\delta_1\mathscr{Z}_j &= \langle\varphi_1\varphi_2\varphi_3\varphi_4\rangle_j &&\triangleq \overline{1234}.\end{aligned}$$

This notation is not totally standard, but you'll find people including yourself will invent different ways to deal with the more tedious calculations. Remember that $\overline{1}$, for example, is still a functional of j so has a nontrivial functional derivative: $\delta_2\overline{1} = \overline{12}$ and so on.
What happens when we consider instead our new generating functional $\mathscr{V}_j$? Well, after some working:

$$\begin{aligned}\delta_4\delta_3\delta_2\delta_1\ln(\mathscr{Z}_j) &= \delta_4\delta_3\delta_2\mathscr{Z}_j^{-1}\overline{1}\\ &= \delta_4\delta_3\left[-\mathscr{Z}_j^{-2}\overline{2}\cdot\overline{1}+\mathscr{Z}_j^{-1}\overline{12}\right]\\ &\vdots\quad\text{(skip to the end...)}\\ &= \mathscr{Z}_j^{-1}\left[\overline{1234}-\mathscr{Z}_j^{-1}\left(\overline{1}\cdot\overline{234}+\overline{2}\cdot\overline{134}+\overline{3}\cdot\overline{124}+\overline{4}\cdot\overline{123}\right)\right.\\ &\quad -\mathscr{Z}_j^{-1}\left(\overline{12}\cdot\overline{34}+\overline{13}\cdot\overline{24}+\overline{14}\cdot\overline{23}\right)\\ &\quad +2\mathscr{Z}_j^{-2}\left(\overline{1}\cdot\overline{2}\cdot\overline{34}+\overline{1}\cdot\overline{3}\cdot\overline{24}+\overline{1}\cdot\overline{4}\cdot\overline{23}+\overline{2}\cdot\overline{3}\cdot\overline{14}+\overline{2}\cdot\overline{4}\cdot\overline{13}+\overline{3}\cdot\overline{4}\cdot\overline{12}\right)\\ &\quad \left.-2\mathscr{Z}_j^{-3}\overline{1}\cdot\overline{2}\cdot\overline{3}\cdot\overline{4}\right].\end{aligned}$$

When we now send j to zero, and note that a choice of $\langle\varphi\rangle = 0$ is sensible under normal circumstances (not in the Higgs field...), and return to the normally streamlined notation, we have

$$\delta_4\delta_3\delta_2\delta_1\ln(\mathscr{Z}_j) = \langle\varphi_1\varphi_2\varphi_3\varphi_4\rangle - \left(\langle\varphi_1\varphi_2\rangle\langle\varphi_3\varphi_4\rangle + \langle\varphi_1\varphi_3\rangle\langle\varphi_2\varphi_4\rangle + \langle\varphi_1\varphi_4\rangle\langle\varphi_2\varphi_3\rangle\right)$$

that is, using $\mathscr{V}$ instead of $\mathscr{W}$ gives exactly the same result but with the disconnected diagrams removed. The result is given diagrammatically in Figure 11.

4 Renormalization

This topic is one of the cornerstones of QFT. In fact we've done a lot of the work already. The classic references cover a lot of this as well [4, 2, 5], in particular Chapter 33 of [3]. The reader interested in the discussion of asymptotic series in Section 4.1 could read the book by Dingle available for free on Michael Berry's website [6], or indeed any number of papers by Berry himself. Mode-Mode Coupling, in Section 4.3, is a specific example of a more general idea and is not of wide relevance in itself.

4.1 Motivation: Asymptotic Series

> *An infinite number of mathematicians walk into a pub. The first orders a pint. The second orders a half-pint, the third orders a quarter-pint, and so on. After a while the Landlord says "I see where this is going. Here's two pints: sort it out amongst yourselves."*

Say we want to find the infinite sum

$$S = 1 + r + r^2 + r^3 + \dots$$

the trick is to use the fact that subtracting one and dividing by r returns the original sum again:

$$\begin{aligned} \frac{S-1}{r} &= 1 + r + r^2 + r^3 + \dots \\ &\equiv S \end{aligned}$$

therefore, rearranging,

$$\begin{aligned} S(1-r) &= 1 \\ S &= \frac{1}{1-r}. \end{aligned}$$

Hence for $r = \frac{1}{2}$, $S = 2$, and the Landlord was correct.

By all rights we can only expect the method to work up to the radius of convergence of the series, $|r| = 1$. To go beyond the radius of convergence requires the techniques of analytic continuation. However, should you choose to sum the series with these techniques (*e.g.* Borel summation) for, say, $r = 2$, you in fact get the same result: in that case, $S = -1$. Summing divergent series is possible, and in fact it's often faster than summing convergent series.

> *An infinite number of mathematicians walk into a pub. The first orders a pint. The second orders two pints, the third four pints, and so on. After a while the Landlord says "I see where this is going. You owe me a pint."*

The relevance to QFT is that the sum of the infinite series of Feynman diagrams is not actually convergent. In fact it is not divergent either; it is known as an Asymptotic Series. It converges for some time, then begins to diverge. There is an excellent book available on asymptotic series on Michael Berry's website [6]. There's an important question as to where we should truncate the series, since adding up as many terms as possible doesn't even necessarily get a better answer. For *practical* QFT matters the options are generally these:

- truncate at zeroeth order (mean field theory)
- truncate at first order (quantum corrections)
- truncate at second order (more quantum corrections - often very tricky)
- sum a subset of the diagrams to infinity
- truncate the series at a physically relevant point.

We have already dealt with the first three points, at least in principle. In the following sections I will deal with the final two.

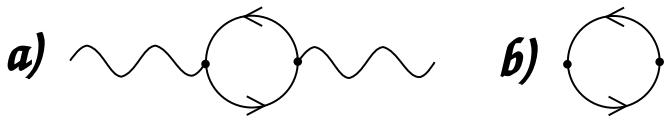

Figure 12: (a)The first order quantum correction to the phonon propagator with an electron-phonon coupling $g\varphi\psi^{\dagger}\psi$. (b) Upon amputating the external legs (again, apologies for the macabre phraseology) the resulting diagram is known as the electronic susceptibility.

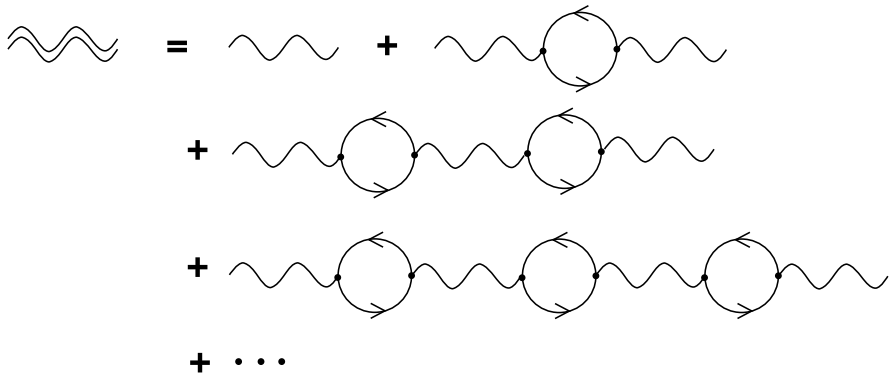

Figure 13: The RPA renormalization of the phonon propagator with the electron-phonon coupling. The infinite series of diagrams includes no correlations between internal electron-hole loops.

4.2 The Random Phase Approximation

The Random Phase Approximation, or RPA, is a method of summing a certain kind of Feynman diagram to infinity. The name generally refers to problems with electron-phonon coupling, but the QED electron-photon vertex is almost identical.

Say we want to change the phonon propagator by adding contributions from higher-order diagrams. Physically we would like to include some effect of the electrons. This is a form of 'renormalization'. The first-order correction, of order g^2, is shown in Figure 12. If we amputate the external phonon lines the resulting electron-hole loop[11] is known as the electronic susceptibility χ.

If we define the bare (unrenormalized) phonon propagator with '4-momentum' $q = (\Omega, \mathbf{q})$ to be $D_0(q)$, and the electron propagator $G(k)$, the expression is

$$\begin{aligned} D(q) &= D_0(q) + g^2 D_0(q) \left[\sum_k G(k) G(k+q) \right] D_0(q) \\ &= D_0(q) + g^2 D_0(q) \chi(q) D_0(q) . \end{aligned}$$

We can keep adding these loops, provided we ignore correlations between the internal electron lines which are doing the renormalization. The electrons' phases are thus in some sense 'randomized', although I think the name Random Phase Approximation is mostly an historical artefact. The diagrams are shown in Figure 13.
The resulting series is then (dropping q dependence for convenience)

$$D = D_0 + g^2 D_0 \chi D_0 + g^4 D_0 \chi D_0 \chi D_0 + g^6 D_0 \chi D_0 \chi D_0 \chi D_0 + \ldots$$

and we can do the sum to infinity using the Landlord's technique:

$$\begin{aligned} g^{-2} \chi^{-1} D_0^{-1} (D - D_0) &= D_0 + g^2 D_0 \chi D_0 + g^4 D_0 \chi D_0 \chi D_0 + \ldots \\ &\equiv D \end{aligned}$$

where in general the propagators may be complicated matrices or tensors so cannot be assumed to commute. Re-arranging, the result is

[11] As before, we could equally interpret this an electron-electron loop and so on; the most common convention here is to interpret the diagram as a phonon breaking up into an electron and hole which then recombine.

Figure 14: The next-order (*i.e.* g^4) diagrams above RPA, known as the mode mode coupling diagrams. A physical reason for going beyond RPA might be that electron correlations (mediated by the phonon field) are of particular importance.

$$D = \left(1 - g^2 D_0 \chi\right)^{-1} D_0$$

which is the RPA approximation for the infinite renormalization of the phonon propagator. The phonon propagator in momentum space takes the form (bare frequency $\Omega_0(\mathbf{q})$ and 'Matsubara frequency' Ω_n):

$$D_0(q) = \frac{-2\Omega_0}{(i\Omega_n)^2 - \Omega_0^2}$$

so in this case the RPA renormalized propagator reads

$$\begin{aligned} D &= \frac{-2\Omega_0}{(i\Omega_n)^2 - \Omega_0^2 + 2\Omega_0 g^2 \chi} \\ &= \frac{-2\Omega_0}{(i\Omega_n)^2 - \Omega_{RPA}^2} \end{aligned}$$

where the RPA renormalized phonon frequency is given by

$$\Omega_{RPA}^2(\mathbf{q}) = \Omega_0^2(\mathbf{q}) - 2g^2\Omega_0(\mathbf{q})\chi(\mathbf{q}).$$

The bare phonon frequency had some normalization, and the process of adding quantum corrections changed it - hence, renormalization. Renormalization goes a lot further than what we'll see in these notes, and in fact plays an important part in allowing QFTs to generate accurate results even though they're mathematically poorly-defined.

4.3 Mode Mode: Beyond RPA

The RPA works very well in many cases, as might be expected of an infinite sum. The final option, on my list of practical ways to truncate Feynman diagram series, was to truncate at a physically relevant point. I will give as an example a case where I've needed to go beyond RPA, still in a theory with electron-phonon coupling. In this case we were arguing that the phonon field was driving a phase transition, and the contributions of the phonons were vital. The lowest order in which phonons influence the susceptibility itself is second order, and the corresponding diagrams are known as the Mode-Mode Coupling approximation, MMA. They are shown in Figure 14.

Rather than go through the details of the calculation I will simply show the effect on the renormalization of the phonon frequency in Figure 15. It can be seen that the RPA renormalizes the frequency down; when the renormalized frequency hits zero the material becomes structurally unstable and a phase transition occurs (it costs zero energy to excite a phonon of that frequency, so the corresponding lattice distortion is permanent). However, the MMA diagrams bring the frequency back up again. Physically the diagrams correspond to fluctuations of the phonon field, which make it harder for the phonons to introduce order.

As a final, philosophical, note on renormalization, it is the renormalized quantities we actually deal with in the real world. The bare propagators are mathematical abstractions, ideal particles which are completely undisturbed by the rest of the universe. Sometimes it is necessary to infer their properties from the particles we see and, as usual, we begin to encounter infinities. To take a popular example which is part analogy, part reality (maybe 'fable' is best): the electric field of an electron diverges as you approach it even in the classical theory

$$E(r) = \frac{Q}{r^2}$$

suggesting an infinite self-energy at $r = 0$. The fable tells us that the charge is screened by vacuum polarization: the QED vertex allows renormalization of quantities such as the observed charge, and the measured result includes the contributions of the renormalization. It's misleading to now start talking about Feynman diagrams since these

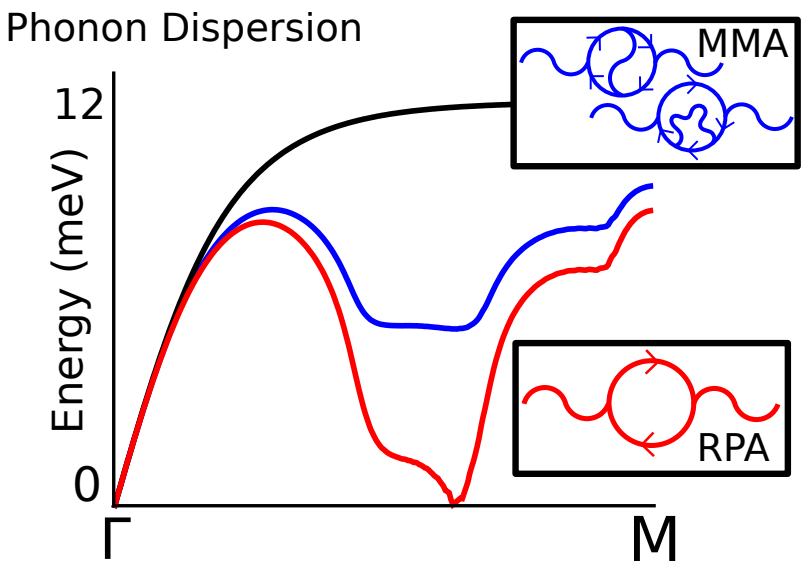

Figure 15: This plot, showing theoretical results on the material $NbSe_2$, shows the physical effect of the different diagrams to the phonon energy. The plot is against momentum-transfer by the phonons, q, along a particular high-symmetry direction in the Brillouin zone. The RPA diagrams lower the phonon energy towards a phase transition, but the MMA diagrams negate the effect to some extent by disordering the system through phonon fluctuations.

are perturbative and you'd want to sum the full infinite series to get a real result - it's even more misleading to start interpreting the diagrams in terms of virtual particles, so we'll stay away from the discussion. What can be said is that the infinity you started from is quite unphysical; the only measurable quantity, the renormalized quantity, is finite.

4.4 Dyson Series: Mass Renormalization

The RPA result of Section 4.2 is actually a special case of a trick which works for arbitrary interactions and arbitrary propagators. I will pick for definiteness the Klein-Gordon propagator

$$D_0(p) = \frac{-1}{p^2 - m^2}. \tag{5}$$

Note that the propagator diverges whenever $p^2 = m^2$, *i.e.* when $E^2 = \mathbf{p}^2 + m^2$. Normally we would always expect this to be true, but internal lines are not required to obey this condition. When they do they are said to be 'on-shell'. It is a general property of propagators that they contain a divergence (actually a 'pole' in the complex p plane) corresponding to the mass of the particle they represent.
In general the corrections to the bare propagator are known as its self-energy, $\Sigma(p)$. These can be more general than the RPA, although this is a particular form of self-energy correction. We can still define a perturbation series with self-energy corrections just as in Section 4.2:

$$D = D_0 + D_0\Sigma D_0 + D_0\Sigma D_0\Sigma D_0 + D_0\Sigma D_0\Sigma D_0\Sigma D_0 + \dots$$

which in this general case is known as a Dyson series. Rearranging with the Landlord's technique (I can now reveal that the Landlord was in fact Freeman Dyson):

$$\begin{aligned} D &= (1 - D_0\Sigma)^{-1} D_0 \\ &= \frac{1}{D_0^{-1} - \Sigma}. \end{aligned}$$

Substituting the specific bare propagator we find in this case that

$$D(p) = \frac{-1}{p^2 - m^2 + \Sigma}. \tag{6}$$

Comparing the bare and renormalized propagators (Equations 5 and 6) we see that the self-energy has shifted the location of the pole to $m^2 - \Sigma$. Since the pole dictates the mass of the particle this means the self-energy has renormalized the mass. This is a general result: except in special cases, self-interactions of a particle shift its mass.

A brief note on QED: the same reasoning might suggest the mass of the photon may renormalize under interaction with electrons, yet we know the renormalized photons we see are truly massless. In some cases symmetries protect the propagators from renormalizing, or rather the renormalization causes no effect. This is the case with the photon propagator, which is protected by a 'Ward identity'. Ward identities are an important feature of QFTs, and amount to making the theory gauge invariant. We won't have time to go through them in this course.

5 Removing and Introducing Fields

In this last section I will introduce two important tricks. The first is how to approximate a theory of multiple fields by removing one of them, arriving at an 'effective action'. This is a very common procedure mathematically, and gets to the heart of QFT philosophically. The second is a neat mathematical trick for introducing new fields which proves of use with complicated interactions.

I'm not familiar with an especially good reference for Section 5.1 so the standard references will have to do their usual good job [4, 2, 5]. I have some excellent notes written by A. Melikyan and M. R. Norman on the Hubbard-Stratanovitch transformation in Section 5.2. They are not available online so please email me if you would like a copy.

5.1 Integrating Out Fields: the Effective Action

Say we have a QFT which contains two independent fields. I will use the QED-type coupling, but for two real scalar fields[12]. This is sometimes referred to as scalar-QED:

$$\mathscr{W} = \mathscr{Z}_\varphi^{-1}\mathscr{Z}_\eta^{-1}\int \mathscr{D}\varphi\mathscr{D}\eta \exp\left(iS_\varphi^0 + iS_\eta^0 + iS_{int}\left[\varphi,\eta\right]\right)$$

where the non-interacting partition functions for the two fields are $\mathscr{Z}_\varphi$ and $\mathscr{Z}_\eta$, and

$$\begin{aligned} S_\varphi^0\left[\varphi\right] &\triangleq \int \mathrm{d}^4x \int \mathrm{d}^4y \left(-\frac{1}{2}\varphi\left(x\right) A\left(x,y\right)^{-1}\varphi\left(y\right)\right) \\ S_\eta^0\left[\eta\right] &\triangleq \int \mathrm{d}^4x \int \mathrm{d}^4y \left(-\frac{1}{2}\eta\left(x\right) B\left(x,y\right)^{-1}\eta\left(y\right)\right) \\ S_{int}\left[\varphi,\eta\right] &\triangleq \int \mathrm{d}^4x \left(g\varphi\left(x\right)\eta\left(x\right)\eta\left(x\right)\right) \end{aligned}$$

so the propagators for the fields are A and B. I've used the symbol $\mathscr{W}$ for the interacting partition function as before.

To *integrate out* one of the fields is to find an effective action featuring only the other field. This requires us to approximate one of the theories to Gaussian form and perform the respective functional integral. First we will integrate out the η field.

Expanding the interaction term as a Taylor series, reverting to the shorter notation $\varphi\left(x\right) = \varphi_x$ *etc.* the partition function is

$$\begin{aligned} \mathscr{W} &= \mathscr{Z}_\varphi^{-1}\mathscr{Z}_\eta^{-1}\int \mathscr{D}\varphi\mathscr{D}\eta \exp\left(-i\frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \left(\varphi_x A_{xy}^{-1}\varphi_y + \eta_x B_{xy}^{-1}\eta_y\right) + ig\int \mathrm{d}^4x\varphi_x\eta_x\eta_x\right) \\ &= \mathscr{Z}_\varphi^{-1}\mathscr{Z}_\eta^{-1}\int \mathscr{D}\varphi\mathscr{D}\eta \exp\left(iS_\varphi\right)\exp\left(iS_\eta\right)\left(1 + ig\int \mathrm{d}^4x\varphi_x\eta_x\eta_x\right. \\ &\quad + \frac{1}{2!}\left(ig\right)^2\int \mathrm{d}^4x \int \mathrm{d}^4y\varphi_x\eta_x\eta_x\varphi_y\eta_y\eta_y \\ &\quad \left. + \frac{1}{3!}\left(ig\right)^3\int \mathrm{d}^4x \int \mathrm{d}^4y \int \mathrm{d}^4z\varphi_x\eta_x\eta_x\varphi_y\eta_y\eta_y\varphi_z\eta_z\eta_z + \ldots\right). \end{aligned}$$

By the definition of the functional average we're hopefully getting quite used to by now this is equivalent to

$$\begin{aligned} \mathscr{W} &= \mathscr{Z}_\varphi^{-1}\int \mathscr{D}\varphi \exp\left(iS_\varphi\right)\left(1 + ig\int \mathrm{d}^4x\varphi_x\left\langle\eta_x\eta_x\right\rangle_\eta\right. \\ &\quad + \frac{1}{2!}\left(ig\right)^2\int \mathrm{d}^4x \int \mathrm{d}^4y\varphi_x\varphi_y\left\langle\eta_x\eta_x\eta_y\eta_y\right\rangle_\eta \\ &\quad \left. + \frac{1}{3!}\left(ig\right)^3\int \mathrm{d}^4x \int \mathrm{d}^4y \int \mathrm{d}^4z\varphi_x\varphi_y\varphi_z\left\langle\eta_x\eta_x\eta_y\eta_y\eta_z\eta_z\right\rangle_\eta\right) \end{aligned}$$

[12] Although I don't know of a physical system to which this theory applies it avoids the complications of gauge fields and Grassman fields associated with photons and electrons respectively. This course is practical in terms of *methods* rather than applications!

with the η subscript indicating the functional average with respect to that field. The results are

$$\begin{aligned}\mathscr{W} &= \mathscr{Z}_\varphi^{-1}\int\mathscr{D}\varphi\exp\left(iS_\varphi\right)\left(1+ig\int\mathrm{d}^4x\varphi_xB_{xx}\right.\\&+\frac{1}{2!}\left(ig\right)^2\int\mathrm{d}^4x\int\mathrm{d}^4y\varphi_x\varphi_y\left(B_{xx}B_{yy}+2B_{xy}^2\right)\\&+\frac{1}{3!}\left(ig\right)^3\int\mathrm{d}^4x\int\mathrm{d}^4y\int\mathrm{d}^4z\varphi_x\varphi_y\varphi_z\left(B_{xx}B_{yy}B_{zz}\right.\\&\left.\left.+2B_{xx}B_{yz}^2+2B_{yy}B_{xz}^2+2B_{zz}B_{xy}^2+4B_{xy}B_{yz}B_{zx}\right)\right)\end{aligned}$$

or, rewriting slightly,

$$\begin{aligned}\mathscr{W} &= \mathscr{Z}_\varphi^{-1}\int\mathscr{D}\varphi\exp\left(iS_\varphi\right)\left(1+ig\int\mathrm{d}^4x\varphi_xB_{xx}+\frac{1}{2!}\left(ig\int\mathrm{d}^4x\varphi_xB_{xx}\right)^2\right.\\&+\frac{1}{3!}\left(ig\int\mathrm{d}^4x\varphi_xB_{xx}\right)^3+\frac{2}{2!}\left(ig\right)^2\iint\mathrm{d}^4x\mathrm{d}^4y\varphi_x\varphi_yB_{xy}^2\\&++\frac{4}{3!}\left(ig\right)^3\int\mathrm{d}^4x\int\mathrm{d}^4y\int\mathrm{d}^4z\varphi_x\varphi_y\varphi_zB_{xy}B_{yz}B_{zx}\\&\left.+\frac{6}{3!}\left(ig\right)^3\int\mathrm{d}^4x\int\mathrm{d}^4y\int\mathrm{d}^4z\varphi_x\varphi_y\varphi_zB_{xx}B_{yz}^2+\ldots\right).\end{aligned}$$

As the first four terms suggest, the series is equivalent to the Taylor series expansion of an exponential to third order; in fact this is true to all orders in the expansion, and allows us to 're-exponentiate':

$$\begin{aligned}\mathscr{W} &= \mathscr{Z}_\varphi^{-1}\int\mathscr{D}\varphi\exp\left(iS_\varphi\right)\exp\left(ig\int\mathrm{d}^4x\varphi_xB_{xx}+i^2g^2\iint\mathrm{d}^4x\mathrm{d}^4y\varphi_x\varphi_yB_{xy}^2\right.\\&\left.+\left(ig\right)^3\int\mathrm{d}^4x\int\mathrm{d}^4y\int\mathrm{d}^4z\varphi_x\varphi_y\varphi_z\left(B_{xx}B_{yz}^2+\frac{2}{3}B_{xy}B_{yz}B_{zx}\right)+\mathcal{O}\left(g^4\right)\right).\end{aligned}$$

The linear and cubic terms disappear in this case if we assume the amplitude to propagate nowhere, B_{xx}, is zero[13]. The linear term is in fact such a common occurrence it gets its own name: the Tadpole diagram (draw it to see why)[14]. You will encounter many pieces of flotsam and jetsam like this when navigating the Dirac sea, and in practical applications they are discarded immediately. Terms of this form cannot conserve momentum. In general, though, higher order terms in the series will lead to higher order interaction terms in the *effective action*, where

$$\mathscr{W}\approx\mathscr{Z}_\varphi^{-1}\int\mathscr{D}\varphi\exp\left(iS_{eff}\left[\varphi\right]\right)$$

that is, an approximation has been made in which the η field has been eliminated. Examining the form for this specific interaction, to this order, we see

$$\begin{aligned}S_{eff} &= \int\mathrm{d}^4x\mathrm{d}^4y\left(-\frac{1}{2}\varphi_xA_{xy}^{-1}\varphi_y+ig^2\varphi_x\varphi_yB_{xy}^2\right)+\frac{2}{3}\left(ig\right)^3\int\mathrm{d}^4x\int\mathrm{d}^4y\int\mathrm{d}^4z\varphi_x\varphi_y\varphi_zB_{xy}B_{yz}B_{zx}+\mathcal{O}\left(g^4\right)\\&= -\frac{1}{2}\mathrm{d}^4x\mathrm{d}^4y\varphi_x\left(A_{xy}^{-1}-2ig^2B_{xy}^2\right)\varphi_y+\frac{2}{3}\left(ig\right)^3\int\mathrm{d}^4x\int\mathrm{d}^4y\int\mathrm{d}^4z\varphi_x\varphi_y\varphi_zB_{xy}B_{yz}B_{zx}+\mathcal{O}\left(g^4\right)\end{aligned}$$

so the lowest order effect here is a simple renormalization of the free φ propagator. The next order term introduces a cubic interaction between φ particles.

The effective action is ubiquitous in applications of QFT. Of particular interest is its application to many modern ideas about the universe, including the standard model of particle physics. The general idea is that fields important at high energy scales may lose relevance on everyday scales, and can be integrated out to give low-energy effective

[13] This isn't always the case, for example if the propagator is a constant.

[14] Lancaster and Blundell [3] note that Sidney Coleman, who invented the name 'tadpole', also suggested 'spermion'.

theories. The particles we see may not be the 'fundamental' particles, whatever they are, but the remains after we've approximated away interactions with other fields we don't deal with directly day-to-day. By way of another fable, think of the Brownian motion of a pollen particle in water. The particle drifts with time - we know this is the result of a random walk caused by individual water molecules colliding with the particle. We can see the pollen move but not the water molecules responsible; they're too 'high energy' in the sense of being too fast-moving and too small (recall $[Length] = \mathbb{E}^{-1}$ so small $\sim$ high energy, where '$\sim$' means 'you get the general idea'). We can construct an effective theory for the motion by integrating out the 'water molecule field' and just maintaining pollen, whose motion is renormalized by the process.

5.2 Inserting New Fields: the Hubbard Stratanovitch Transformation

Our final useful trick allows us to simplify certain types of complicated interaction at the expense of introducing a new field. The new field can often then be ascribed a physical interpretation. As we're familiar with the manipulations of QFT at this stage I'm going to break my one rule and move away from real scalar fields, introducing instead fermionic (anticommuting) *Grassman* fields. The reason I think this is okay here is that the new fields will couple into a real bosonic field, meaning their anticommuting property does not complicate things.

This time the action describes electrons on a lattice. The field operator $\hat{\psi}^\dagger_{\sigma i}(t)$ creates an electron with spin $\sigma \in \{\uparrow, \downarrow\}$ on site i. As we're on a lattice there's no Lorentz invariance and the operators carry a time label t. We will introduce a repulsive interaction between electrons on the same site[15]:

$$S\left[\hat{\psi}^\dagger, \hat{\psi}\right] = \int \mathrm{d}t \sum_i \hat{\psi}^\dagger_i G_i^{-1} \hat{\psi}_i + g \sum_i \hat{\psi}^\dagger_{\uparrow i} \hat{\psi}_{\uparrow i} \hat{\psi}^\dagger_{\downarrow i} \hat{\psi}_{\downarrow i}.$$

Just like in the harmonic oscillator in ordinary QM the operator $\hat{n}_i \triangleq \hat{\psi}^\dagger_i \hat{\psi}_i$ is the number operator. Now we're working with fields there is a different one for each site i. The Pauli exclusion principle disallows double-occupation of the same site if the electrons have the same spin, *i.e.* $\hat{n}_{\sigma i}\hat{n}_{\sigma i} \equiv 0$.

First we recall the definition of the free partition function, this time for a field with a constant propagator g:

$$\mathscr{Z}_\Delta = \int \mathscr{D}\hat{\Delta} \exp\left(-\frac{i}{2} g \int \mathrm{d}t \sum_i \hat{\Delta}_i^2\right).$$

In an ordinary infinite integral $\int_{-\infty}^{\infty} \mathrm{d}x$ we are free to change variables to $x' = x + a$ and leave the limits unchanged. The same applies to the functional integral, which is effectively always 'infinite', so we can shift the field to

$$\begin{aligned} \hat{\Delta}'_i &= \hat{\Delta}_i + \sum_\sigma \hat{\psi}^\dagger_{\sigma i} \hat{\psi}_{\sigma i} \\ \mathscr{D}\hat{\Delta}' &= \mathscr{D}\hat{\Delta} \end{aligned}$$

giving

$$\mathscr{Z}_\Delta = \int \mathscr{D}\hat{\Delta}' \exp\left(-\frac{i}{2} g \int \mathrm{d}t \sum_i \left(\hat{\Delta}'_i - \sum_\sigma \hat{\psi}^\dagger_{\sigma i} \hat{\psi}_{\sigma i}\right)\left(\hat{\Delta}'_i - \sum_\sigma \hat{\psi}^\dagger_{\sigma i} \hat{\psi}_{\sigma i}\right)\right)$$

and expanding (also relabel Δ' back to Δ)

$$\mathscr{Z}_\Delta = \int \mathscr{D}\hat{\Delta} \exp\left(-\frac{i}{2} g \int \mathrm{d}t \sum_i \hat{\Delta}_i \hat{\Delta}_i\right) \exp\left(ig \int \mathrm{d}t \sum_{\sigma i} \hat{\Delta}_i \hat{\psi}^\dagger_{\sigma i} \hat{\psi}_{\sigma i}\right) \exp\left(-\frac{i}{2} g \int \mathrm{d}t \sum_{\sigma\sigma' i} \hat{\psi}^\dagger_{\sigma i} \hat{\psi}_{\sigma i} \hat{\psi}^\dagger_{\sigma' i} \hat{\psi}_{\sigma' i}\right)$$

finally, noting that if $\sigma = \sigma'$ Pauli exclusion sends the quartic term to zero, then rearranging slightly we see

$$\exp\left(ig \int \mathrm{d}t \sum_i \hat{\psi}^\dagger_{\uparrow i} \hat{\psi}_{\uparrow i} \hat{\psi}^\dagger_{\downarrow i} \hat{\psi}_{\downarrow i}\right) = \mathscr{Z}_\Delta^{-1} \int \mathscr{D}\hat{\Delta} \exp\left(-\frac{i}{2} g \int \mathrm{d}t \sum_i \hat{\Delta}_i \hat{\Delta}_i\right) \exp\left(ig \int \mathrm{d}t \sum_{\sigma i} \hat{\Delta}_i \hat{\psi}^\dagger_{\sigma i} \hat{\psi}_{\sigma i}\right).$$

[15] Unlike scalar fields the Grassman field doesn't need a $-\frac{1}{2}$ normalization for the propagator.

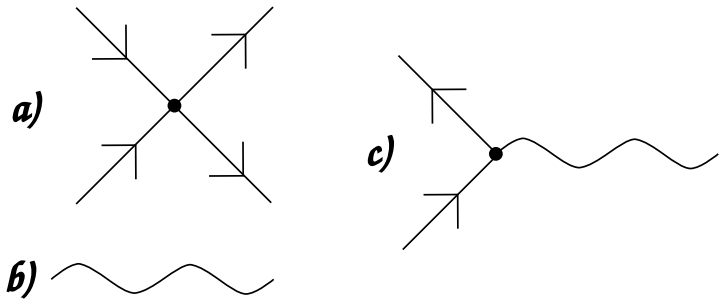

Figure 16: The Hubbard Stratanovitch transformation removes a 4-fermion vertex, (a), at the cost of introducing a real bosonic field (b) and an interaction between the two, (c).

This is precisely what we need to eliminate the quartic term from the partition function of the original theory:

$$\begin{aligned}
\mathscr{Z} &= \mathscr{Z}_\psi^{-1} \int \mathscr{D}\hat{\psi}^\dagger \mathscr{D}\hat{\psi} \exp\left(i \int \mathrm{dt} \sum_i \hat{\psi}_i^\dagger G_i^{-1} \hat{\psi}_i + ig \int \mathrm{dt} \sum_i \hat{\psi}_{\uparrow i}^\dagger \hat{\psi}_{\uparrow i} \hat{\psi}_{\downarrow i}^\dagger \hat{\psi}_{\downarrow i} \right) \\
&= \mathscr{Z}_\Delta^{-1} \mathscr{Z}_\psi^{-1} \int \mathscr{D}\hat{\psi}^\dagger \mathscr{D}\hat{\psi} \mathscr{D}\hat{\Delta} \exp\left(i \int \mathrm{dt} \sum_i \hat{\psi}_i^\dagger G_i^{-1} \hat{\psi}_i \right) \exp\left(-\frac{i}{2} g \int \mathrm{dt} \sum_i \hat{\Delta}_i \hat{\Delta}_i \right) \exp\left(ig \int \mathrm{dt} \sum_{\sigma i} \hat{\Delta}_i \hat{\psi}_{\sigma i}^\dagger \hat{\psi}_{\sigma i} \right) \\
&= \mathscr{Z}_\Delta^{-1} \mathscr{Z}_\psi^{-1} \int \mathscr{D}\hat{\psi}^\dagger \mathscr{D}\hat{\psi} \mathscr{D}\hat{\Delta} \exp\left(i S_{HS}\left[\hat{\psi}^\dagger, \hat{\psi}, \hat{\Delta}\right] \right)
\end{aligned}$$

where the Hubbard-Stratanovitch action is given by

$$S_{\mathrm{HS}}\left[\hat{\psi}^\dagger, \hat{\psi}, \hat{\Delta}\right] = \int \mathrm{dt} \sum_i \hat{\psi}_i^\dagger G_i^{-1} \hat{\psi}_i - \frac{1}{2} g \int \mathrm{dt} \sum_i \hat{\Delta}_i \hat{\Delta}_i + g \int \mathrm{dt} \sum_{\sigma i} \hat{\Delta}_i \hat{\psi}_{\sigma i}^\dagger \hat{\psi}_{\sigma i}.$$

Thus we see that we have swapped a quartic interaction between electrons for a real bosonic field $\hat{\Delta}$ with a trivial propagator, and our old friend the QED-type interaction vertex. The situation is shown diagrammatically in Figure 16.

It is important to note that the equivalence between the theories is exact - no approximations have entered at all. One use of the transformation is that the expectation value of the $\hat{\Delta}$ field, $\Delta = \left\langle \hat{\Delta} \right\rangle$, has a natural interpretation as a Cooper pair order parameter. This is easiest seen by noting that $\hat{\Delta}$ took the place of $\hat{\psi}_\sigma^\dagger \hat{\psi}_\sigma$ in a somewhat approximate sense.

5.3 Epilogue

After discussion with various friends and colleagues I feel it necessary to mention some of the concerns they have raised. First I should make clear that this introduction has assumed 'perturbative' QFT. The most up-to-date research goes beyond perturbative methods in various ways. For an important recent discovery in this direction look up 'amplitudihedrons'. Contrary to my opening claim that "The key object in a QFT is the action", there exist QFTs which have no Lagrangian description, particularly in M-theory. Numerical QFT, including lattice QCD, has long been able to consider theories away from Gaussian fixed points. Renormalization is the more modern way to view QFT, and provides an explanation and more rigorous justification for many features which appeared to work without our understanding, contradicting my claims in the introduction. In particular Haag's theorem, that one cannot consistently define the vacuum in interacting theories, is now considered to have been the wrong way to look at the problem, and largely disappears with considerations of RG flow.