Towards a Physical Analogue of a Gott Time Machine

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1 Gravity in $2+1$ Dimensions

General relativity, in its most basic form, consists of finding solutions to the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{1}$$

If we lived in Abbott's *flatland* with only 2 spatial dimensions this task would prove rather simple. The metric associated with a circular mass of radius r_0 in such a world is

$$ds^2 = \begin{cases} dr^2 + (1-4\mu)^2 r^2 d\phi^2 - dt^2, & r \geq r_0 \\ \frac{r_0^2}{(1-4\mu)^{-2}-1} d\phi^2 - dt^2, & r < r_0 \end{cases}$$

from which we see that outside the mass the world truly is 'flat', in the sense that it has zero Gaussian curvature and light rays follow straight lines. To make this more apparent we can change variables to

$$\phi' = (1-4\mu)\,\phi$$

in which case we have

$$ds^2 = dr^2 + r^2 d\phi'^2 - dt^2, \qquad r \geq r_0$$

(the standard Minkowski metric in plane polar co-ordinates) but now the range of the azimuthal co-ordinate is restricted to

$$\phi' \in (0\,,\, 2\pi - 8\pi\mu]\,.$$

From this we see that a mass in $2+1D$ has associated with it a *conical defect* (see Figure 1).

This seemingly innocuous statement has some odd consequences, not least of which is that any two points in the space are connected by two different straight lines (one either side of the mass, or one either side of the apex in the cone picture). This is shown in Figure 2. As a result we get an extreme gravitational lensing effect around any mass in $2+1D$.

The relevance to $3+1D$

Gravity in two spatial dimensions is trivial, with no local degrees of freedom (all freedoms are *topological*). Nonetheless there is potentially direct relevance to our universe if 'cosmic strings' exist. These entities are infinitely long, finite thickness lines of mass predicted by some cosmological models to have been created during the big bang. Their mass per unit length is assumed low enough that they do not form black holes. If a cosmic string stretches along the $\hat{\mathbf{z}}$ axis then for any constant $\mathbf{z}$ we have a $2+1D$ plane containing a mass as described above. The results of these notes can be trivially carried over to our universe provided cosmic strings exist - which could perhaps be considered an argument against their existence.

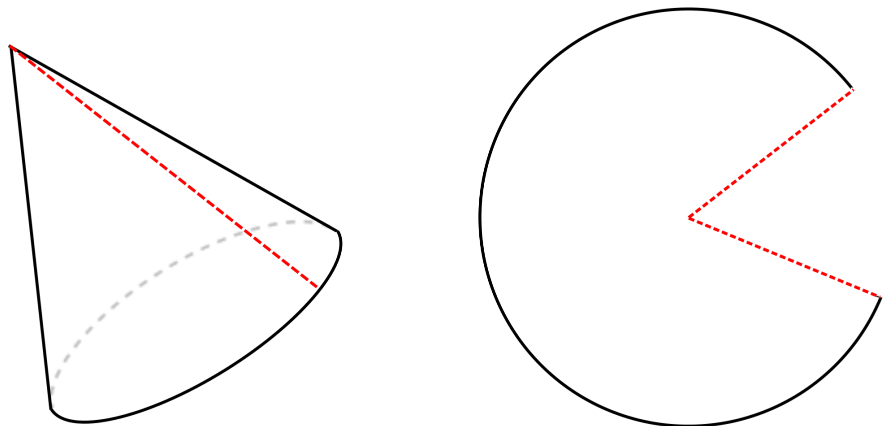


Figure 1: A cone has zero Gaussian curvature away from its apex - it is identical to a flat sheet with a slice removed, and the resulting edges joined.

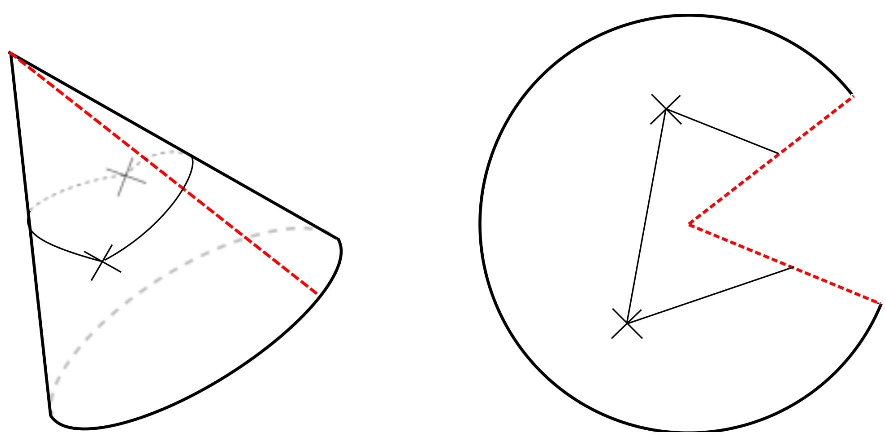


Figure 2: Two different straight lines meet at two points on a cone.

2 The Gott Time Machine

Consider now two identical masses in $2+1D$, located at $(0, \pm d)$. Each has a conical defect. As it is convenient to draw the masses on this flat piece of paper we will use the representation of a cone as a Pacman, with his two jaws describing the same line in the space. Note that the cut introduced in mapping the cone to Pacman is arbitrarily located - there is no physical cut in the space. All that is required is that traversing a closed loop encircling a mass necessarily crosses one cut.

I will now reproduce Gott's analysis, although the original paper [1] is very readable. For $y > 0$ we'll use the following co-ordinates:

$$\begin{aligned} x &= r \sin\left(\phi' + 4\pi\mu\right) \\ y &= r \cos\left(\phi' + 4\pi\mu\right) + d \end{aligned}$$

with metric

$$ds^2 = dx^2 + dy^2 - dt^2$$

and a mirrored situation for $y \leq 0$. The situation is pictured in Figure 3. Pacman's mouth is centred along $y \in [d, \infty)$ in the upper half-plane, so that there's a slice of space missing in this region - if you were to throw a stone towards the line $y = x \tan(4\pi\mu) + d$ from the right it would come straight out of the line $y = x \tan(-4\pi\mu) + d$ as these are really the same line. Recall that the 'cut' is just an artefact of our desire to draw the picture on this sheet of paper, when it's really on a cut-free cone.

Consider two points in space $A = (x_0, 0)$, $B = (-x_0, 0)$. Note that an observer at B sees three copies of A; one from light travelling straight through $A \to B$, one from light travelling through the top cut $A \to C \to B$, and one travelling through the bottom cut $A \to D \to B$. Depending on the angle $4\pi\mu$ the distance ACB can be made smaller than the distance AB, so that light travelling through the cut arrives before light taking the straight-through route. This means a sufficiently fast rocket following ACB can also overtake light following AB.

Say the rocket starts at A at event $E_i = (x_0, 0, -w_0/\beta_R)$ and ends at B at event $(-x_0, 0, w_0/\beta_R)$. Following ACB the distance travelled is $2w_0$, in time $2w_0/\beta_R$. Light following AB takes time $2x_0$, so our rocket arrives first provided

$$x_0 > w_0/\beta_R$$

which is quite possible to fulfil. This is also, by construction, the condition that the events E_i and E_f are spacelike separated. If this is the case it should be possible to boost into a frame in which they occur simultaneously. We do this by giving the mass a velocity β_M along the $+\hat{\mathbf{x}}$ direction. The Lorentz transformations are

$$\begin{aligned} x &= \gamma_M \left(x' + \beta_M t'\right) \\ t &= \gamma_M \left(t' + \beta_M x'\right) \end{aligned}$$

where the primed frame is that of the mass and the unprimed is that of the lab. Thus in the lab frame we now have

$$\begin{aligned} E_i &= \gamma_M \left(x_0 - w_0 \frac{\beta_M}{\beta_R},\, 0,\, -w_0/\beta_R + \beta_M x_0 \right) \\ E_f &= -E_i \end{aligned}$$

and choosing

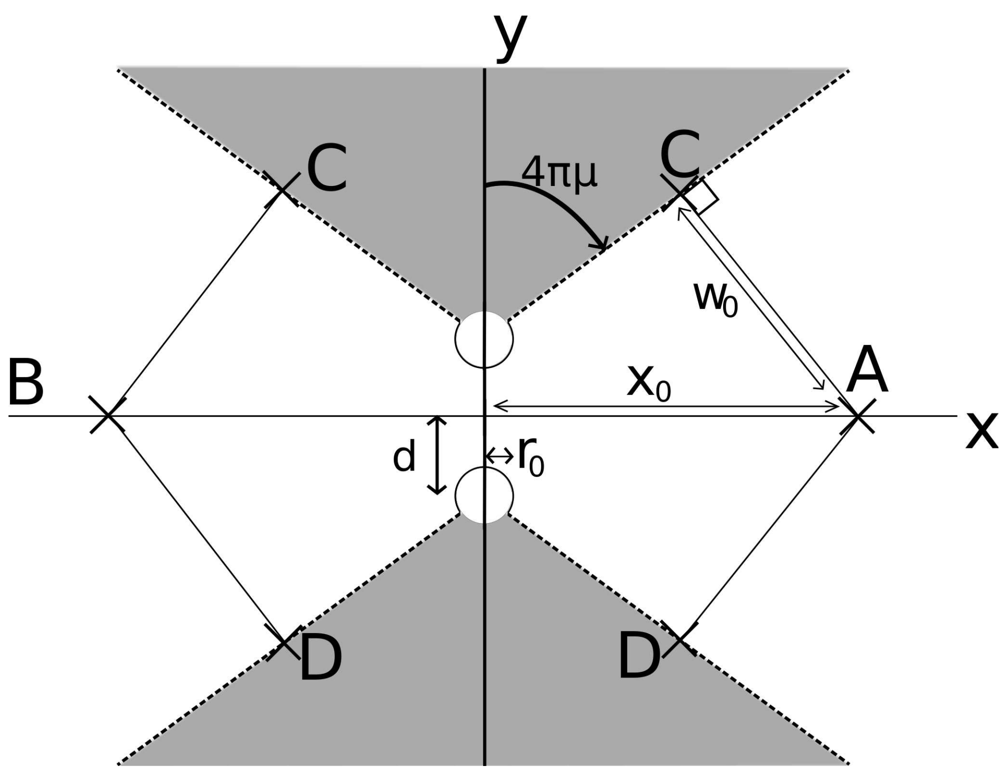


Figure 3: The setup used by Gott [1]. A to D label points in space (points on the cone); as the two edges of each shaded region are equivalent both labels C are for the same point on the cone - the cut is merely an artefact of drawing the cone on a flat page. The path AC is chosen such that it hits the cut at right angles, and for a range of valid choices of d and μ we have that $w_0 < x_0$, meaning light (or a fast enough spaceship) can traverse the route ACB faster than light can traverse the route AB. This is the crux of the time machine's operation.

$$\beta_M = \frac{w_0}{x_0 \beta_R}$$

gives

$$\begin{aligned} E_i &= \frac{\gamma_M}{\beta_R^2 x_0}\left(x_0^2\beta_R^2 - w_0^2\right)\cdot(1,\,0,\,0) \\ E_f &= -E_i \end{aligned}$$

and the events are simultaneous in the lab frame; the rocket leaves A and flies at subluminal speed with no acceleration, and arrives at B at the same time it left!

The really clever part is now to boost the mass located at $(0, -d)$ in the $-\hat{\mathbf{x}}$ direction. The exact analysis we just performed in the upper half plane holds by symmetry, and so the rocket is able to traverse the circuit $ACBDA$, arriving back at A at the same time it left. To make a frame-independent statement, a second event occurs at exactly the same point in spacetime as E_i. The situation is depicted in Figure 4.

The grandfather paradox

What happens if the returning pilot tells themself never to leave? Similarly, what if they go back further in time and kill their own grandfather, are never born, etc. etc.? I propose, by analogy to the condensed matter case of the next section, that the paradox is resolved by being careful about whose grandfather is killed. This is speculative, but I think a full description of the space in the paper-cut metric is actually an n-fold Riemann surface, for conical deficit $2\pi/n$, joined along the cut. Thus the pilot traverses the closed timelike curve and meets their alter-ego on another sheet of the universe, or kills that person's grandfather. This is shown in Figure 5.

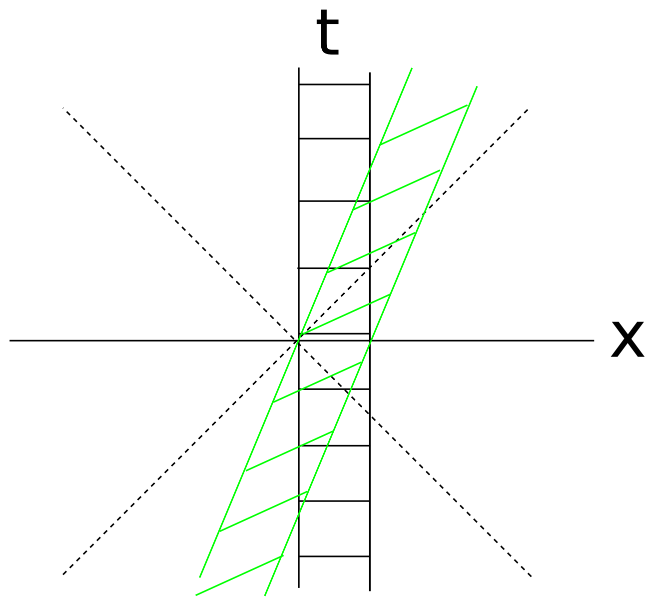
The inspiration is multivalued scattering angles of non-Abelian anyons in condensed matter. When dealing with non-Abelions as they're called, it's important to define a 'bureau of standards' [2]. The problem is that passing one non-Abelion around another causes properties of both (charge, spin etc.) to change. However, the change turns out to be an artefact of not passing one's measuring apparatus along the same path. If that were done no change would be seen. The bureau of standards is a set of bins each containing a different non-Abelion; to see which you have, you pass it along a prescribed route to a bin and compare it to the particle inside. This avoids confusion when it starts to look like the particles are changing by moving around - by sticking to your path and using your bureau you can always check which of the species it really is. The problem is not just a mathematical construct, though - scattering one non-Abelion off another gives multivalued scattering angles (a multivalued observable!).

A specific example is the scattering of charged particles off 'Alice strings' [2]. If a positively charged particle is transported around the string it comes back negatively charged. There are two measurable multivalued cross-sections for the two charge types, $\sigma_+(\theta)$ and $\sigma_-(\theta)$. Their sum is single-valued. The trick is that if you pass your positively-charged-particle-detector around the Alice string it *itself* changes to a negatively-charged-particle-detector.

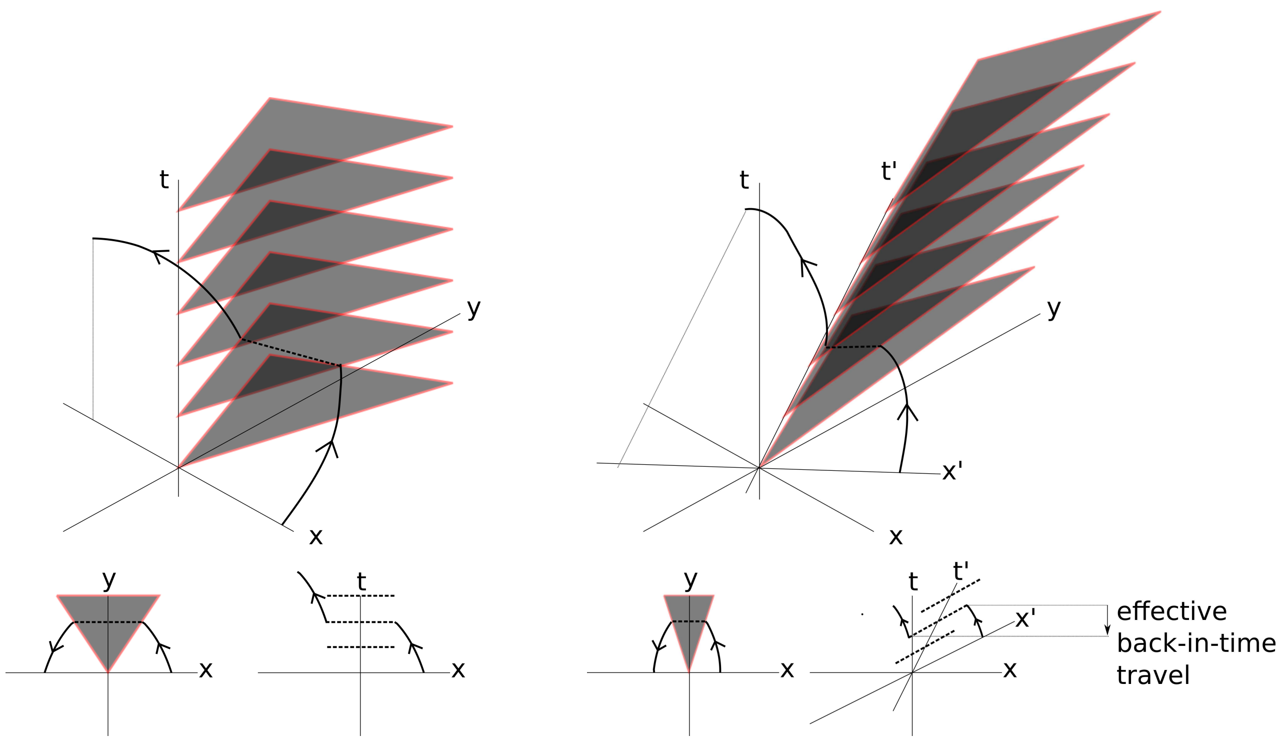
Analogously I propose the solution to the grandfather paradox is that the pilot who carries out the loop can only kill past-grandfathers on other Riemann sheets. They could try grabbing the grandfather they meet and taking them back around the loop in the other direction, back to their cone, then killing them - but this cancels the effect of the time travel.

3 Gravity and Chern Simons Theory

The previous analysis is nice, but suffers from the fact that we don't live in $2+1D$ and so we can't ever test it. Gott's original paper in fact deals with cosmic strings in $3+1D$ and so could in theory be relevant, but of course we've never seen any evidence for cosmic strings, and there would remain a modest experimental barrier to overcome should that change.



(a) Under Lorentz transformations all lines tilt towards the light cone (dashed), so a 'ladder' of events gets skewed. The rungs of the ladder could for example be a line of alarm clocks all set to go off simultaneously every 5 minutes when viewed in their rest frame.



(b) Left: a Minkowski diagram of a mass at the origin, with the corresponding conical defect chosen along the y-axis. Projections are given underneath. The defect at set time intervals appears as a 'toblerone' of spacetime slices. A rocket follows a curved path around the mass (curved here as I feel it makes the picture clearer). Right: the same scenario, but this time the mass is boosted along the x-axis. In the lab frame (unprimed) the ridges of the toblerone tilt as did the rungs of the ladder in (a), causing a lab-based observer to suggest there is a point at which the rocket jumps back in time at a fixed location in space. Note that the cut is arbitrary, so it's not possible for the observer to say where the jump occurs - it's only when the rocket closes the loop in space and returns to its start point that the clock mis-match is found.

Figure 4: Minkowski diagrams explaining the time travel mechanism.

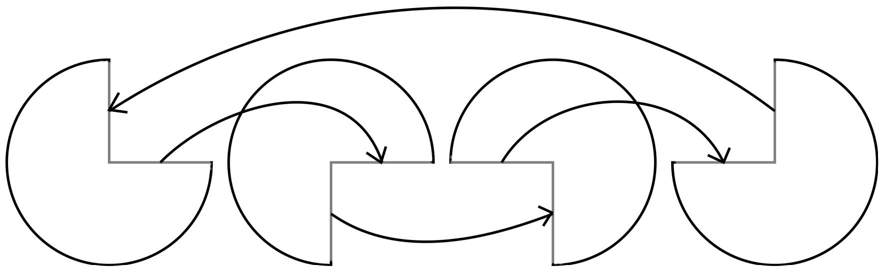


Figure 5: The cone with $2\pi/n$ deficit is really n cones joined along the branch cuts to form an n-fold Riemann surface. Passing through the branch cut moves you onto the next cone. The grandfather paradox is resolved since the closed timelike curve requires crossing two cuts - hence you can only kill the grandfather of your two-cones-away self.

Another possibility is to look to analogue systems. In this section I will address the equivalence of $2+1D$ general relativity to a non-Abelian gauge model known as Chern Simons theory.

Shortly after Einstein wrote down his field equations (Equation 1) it was noticed that they could be written as the Euler Lagrange equation for a field theory. The corresponding action is known as the Einstein Hilbert action, and is given by

$$S_{EH}\left[g\right]=\frac{1}{16\pi}\int \mathrm{d}^3x\sqrt{-\det\left(g\right)}\left(R-2\Lambda\right)$$

in 3 spacetime dimensions (*i.e.* $2+1D$). Varying the action with respect to the metric g results in Equation 1. The form given here includes a (negative) cosmological constant Λ which will prove useful shortly.

The Chern Simons action is given by

$$S\left[A\right]=\frac{k}{4\pi}\int \mathrm{d}^3x\epsilon^{\mu\nu\rho}\mathrm{tr}\left(A^a_\mu\partial_\nu A^a_\rho+\frac{2}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right)$$

where f^{abc} are the group structure constants and k is an integer known as the 'level' of the theory. The field A^a_μ can be written as a 3-vector (by which I mean a 4-vector with only 2 spatial components) with each component a square matrix taking values in the corresponding gauge group. Of interest here is $SU\left(2\right)$ Chern Simons theory in which we can take A^a_μ to be a 3-vector whose elements are linear combinations of the Pauli matrices. In this case the structure factors can be written $f^{abc}=\epsilon^{abc}$. The matrix representation of the group is assumed in the action, where the trace is taken over these matrices.

The relation between the two theories comes from the fact that 2 copies of $SU\left(2\right)$ Chern Simons gives the Einstein Hilbert action for $2+1D$ gravity with a negative cosmological constant. This section follows [3] closely. First, make 2 copies by bumping the group up from $SU\left(2\right)$ to $SL\left(2,\mathbb{C}\right)$, and write the action in terms of a holomorphic and antiholomorphic part:

$$\begin{aligned}S\left[A\right]-S\left[\bar{A}\right] &= \frac{k}{8\pi}\int \mathrm{d}^3x\epsilon^{\mu\nu\rho}\left\{\mathrm{tr}\left(A^a_\mu\partial_\nu A^a_\rho+\frac{2}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right)\right.\\ &\left.-\mathrm{tr}\left(\bar{A}^a_\mu\partial_\nu\bar{A}^a_\rho+\frac{2}{3}f^{abc}\bar{A}^a_\mu\bar{A}^b_\nu\bar{A}^c_\rho\right)\right\}\end{aligned}$$

where $A^a_\mu\in SL\left(2,\mathbb{C}\right)$ and $\bar{A}^a_\mu$ is its compex conjugate. Incidentally, it should be clear from this that $\mathcal{Z}_{SL(2,\mathbb{C})}=\left|\mathcal{Z}_{SU(2)}\right|^2$.

Now change variables to $A^a_\mu=\omega^a_\mu+ie^a_\mu/l$ and $\bar{A}^a_\mu=\omega^a_\mu-ie^a_\mu/l$. The field ω^a_μ is the 'gravitational spin connection'. It can be understood as follows: if we take a field theory valid on a flat manifold and attempt to apply it to a curved manifold, one issue we run into is that our derivatives ∂_μ 'miss' their new target. To patch this up we introduce a 'connection' which connects the point the derivative would have hit to the point it should: $\nabla_\mu = \partial_\mu + iA_\mu$. The same is true of spin-space, and when considering the curved spacetimes of general relativity a connection has to be added to covectors targeting spins. The field e^a_μ is called the dreibein; it's a set of 3 orthogonal axes at each point in space, and can be thought of as a 'square root' of the metric g. The parameter l is a constant.

With this substitution, after a lengthy rearrangement, we obtain the desired result:

$$S\left[A\right] - S\left[\bar{A}\right] = \frac{k}{4\pi l} \int \mathrm{d}^3x \sqrt{-\left|g\right|} \left(R + \frac{2}{l^2}\right)$$

where $k = l/4$ and we need to set $l^{-2} = -\Lambda$.

The point

Chern Simons theory is an effective field theory for anyons. It governs the quasiparticles found in the quantum Hall effect (Abelian) and fractional quantum Hall effect (Abelian or non-Abelian). More pertinently, it is believed that the superconductor Strontium Ruthenate exhibits quasiparticles described by an $SU\left(2\right)$ Chern Simons theory as presented in Section 3.

By using the mapping from $2+1D$ classical gravity to $SU\left(2\right)$ Chern Simons theory it may be possible to map the Gott time machine to observable properties of Strontium Ruthenate. Of course, we wouldn't expect the holonomy to be in the time variable in the material. I suspect the result will rather be a prediction regarding multivalued scattering as presented in Section 2.

In return for the new predictions in the condensed matter system, gravity is given an experimental test of a controversial prediciton of its own.

Acknowledgments

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References

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