# The Chain Rule

March 29, 2012

Here's my solution to the first part of PS4 Q5. Please have a go anyway and hand in your attempts (by Wednesday as usual). I'm hoping you'll be able to do the harder case of the Lorentz transformations using the same method. Please get in touch if you have problems.

## 1 General method

First, the general chain rule result:

$$\left(\frac{\partial u}{\partial x}\right)_t = \left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial u}{\partial x'}\right)_{t'} + \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial u}{\partial t'}\right)_{x'} \tag{1}$$

hopefully you've seen how vital it is that you write what's held constant each time, even in this simple example. We need to know the partial derivatives with respect to $x$ on the right hand side. Our coördinate transformations are as follows:

$$\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned} \tag{2}$$

(Gallilean transformations). So the bits we need are:

$$\begin{aligned} \left(\frac{\partial x'}{\partial x}\right)_t &= 1 \\ \left(\frac{\partial t'}{\partial x}\right)_t &= 0 \end{aligned} \tag{3}$$

and substituting them back into Equation 1 we find

$$\left(\frac{\partial u}{\partial x}\right)_t = \left(\frac{\partial u}{\partial x'}\right)_{t'}. \tag{4}$$

For the $t$ derivative,

$$\left(\frac{\partial u}{\partial t}\right)_x = \left(\frac{\partial t'}{\partial t}\right)_x \left(\frac{\partial u}{\partial t'}\right)_{x'} + \left(\frac{\partial x'}{\partial t}\right)_x \left(\frac{\partial u}{\partial x'}\right)_{t'}$$

and we need the $t$ derivatives on the right hand side. From Equation 2 we find

$$\begin{aligned} \left(\frac{\partial t'}{\partial t}\right)_x &= 1 \\ \left(\frac{\partial x'}{\partial t}\right)_x &= -v \end{aligned} \tag{5}$$

and so

$$\left(\frac{\partial u}{\partial t}\right)_x = \left(\frac{\partial u}{\partial t'}\right)_{x'} - v\left(\frac{\partial u}{\partial x'}\right)_{t'}. \tag{6}$$

Now for the second derivatives. First note that

$$u_{xx} = \left(\frac{\partial^2 u}{\partial x^2}\right)_t = \left(\frac{\partial u_x}{\partial x}\right)_t$$

where, in the second equality, I've mixed my notation writing $u_x = (\partial u/\partial x)_t$. The question really reduces to doing the same thing as in the first part, but with $u \to u_x$. Thus we have

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_t = \left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial u_x}{\partial x'}\right)_{t'} + \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial u_x}{\partial t'}\right)_{x'}$$

and using Equation 3 we get

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_t = \left(\frac{\partial u_x}{\partial x'}\right)_{t'}.$$

Now we substitute our result for the first part, Equation 4, to get

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_t = \left(\frac{\partial^2 u}{\partial x'^2}\right)_{t'}$$

or

$$u_{xx} = u_{x'x'}.$$

The second $t$ derivative follows in the same way:

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_x = \left(\frac{\partial t'}{\partial t}\right)_x \left(\frac{\partial u_t}{\partial t'}\right)_{x'} + \left(\frac{\partial x'}{\partial t}\right)_x \left(\frac{\partial u_t}{\partial x'}\right)_{t'}$$

and substituting Equation 5

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_x = \left(\frac{\partial u_t}{\partial t'}\right)_{x'} - v\left(\frac{\partial u_t}{\partial x'}\right)_{t'}.$$

Now we stick in our first result, Equation 6, to get

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_x = \left(\frac{\partial^2 u}{\partial t'^2}\right)_{x'} - v\left(\frac{\partial^2 u}{\partial t'\partial x'}\right) - v\left(\frac{\partial^2 u}{\partial t'\partial x'}\right) + v^2\left(\frac{\partial^2 u}{\partial x'^2}\right)_{t'}$$

*i.e.*

$$u_{tt} = u_{t't'} - 2vu_{t'x'} + v^2 u_{x'x'}$$

and so the answer is

$$u_{tt} - c^2 u_{xx} = u_{t't'} - 2vu_{t'x'} + \left(v^2 - c^2\right) u_{x'x'}.$$

The point, of course, is that the right hand side is not the wave equation. Umad, Gallileo?

## 2 A faster way

This method's quicker and neater, but requires an understanding of operators. Don't worry about the following it if you're not completely happy with Section 1. If you're a whizz at Section 1 give this method a go.

The trick to operators is that the reasoning in Section 1 didn't require any knowledge of the function $u$. So the results were a property of the $\left(\frac{\partial}{\partial x}\right)_t$ bits, not the $u$ bits. So let's cut out the $u$s. We have

$$\begin{aligned}
\left(\frac{\partial}{\partial x}\right)_t &= \left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial}{\partial x'}\right)_{t'} + \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial}{\partial t'}\right)_{x'} \\
\left(\frac{\partial}{\partial t}\right)_x &= \left(\frac{\partial t'}{\partial t}\right)_x \left(\frac{\partial}{\partial t'}\right)_{x'} + \left(\frac{\partial x'}{\partial t}\right)_x \left(\frac{\partial}{\partial x'}\right)_{t'}.
\end{aligned}$$

We could plug in Equation 2 at this point - and if you want to try this method you should do this to simplify things - but I'll keep the coördinate transformation general for now. So we need the second derivatives. The $x$ case is done by noting that

$$\left(\frac{\partial^2}{\partial x^2}\right)_t = \left(\frac{\partial}{\partial x}\right)_t \left(\frac{\partial}{\partial x}\right)_t = \left(\left(\frac{\partial}{\partial x}\right)_t\right)^2$$

(this wouldn't work if we had left $u$ in, since we'd have a $u^2$ on the right and a $u$ on the left, and our units wouldn't match). Substituting the first result in, we have

$$\left(\frac{\partial^2}{\partial x^2}\right)_t = \left(\left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial}{\partial x'}\right)_{t'} + \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial}{\partial t'}\right)_{x'}\right) \left(\left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial}{\partial x'}\right)_{t'} + \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial}{\partial t'}\right)_{x'}\right).$$

The problem with operators is that they don't commute, so we have to be careful here. In the general case, $\left(\partial x'/\partial x\right)_t$ (for example) will be a function of $t'$ and $x'$, so we have to act on it with the operators in the left parentheses. We also have to remember that

$$\left(\frac{\partial}{\partial x}\right)_t A(x)\,B(x) = \left(\frac{\partial A}{\partial x}\right)_t B + A\left(\frac{\partial B}{\partial x}\right)_t.$$

Expanding, then, gives

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2}\right)_t &= \left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial x'}{\partial x'\partial x}\right) \left(\frac{\partial}{\partial x'}\right)_{t'} + \left(\left(\frac{\partial x'}{\partial x}\right)_t\right)^2 \left(\frac{\partial^2}{\partial x'^2}\right)_{t'} + \left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial^2 t'}{\partial x'\partial x}\right) \left(\frac{\partial}{\partial t'}\right)_{x'} \\ &+ \left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial^2}{\partial x'\partial t'}\right) + \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial x'}{\partial t'\partial x}\right) \left(\frac{\partial}{\partial x'}\right)_t + \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial x'}{\partial x}\right)_t \left(\frac{\partial^2}{\partial t'\partial x'}\right) \\ &+ \left(\frac{\partial t'}{\partial x}\right)_t \left(\frac{\partial^2 t'}{\partial t'\partial x}\right) \left(\frac{\partial}{\partial t'}\right)_{x'} + \left(\left(\frac{\partial t'}{\partial x}\right)_t\right)^2 \left(\frac{\partial^2}{\partial t'^2}\right)\end{aligned}$$

I agree it's a complete mess, but remember it's completely general. Now we can stick in Equations 3 and 5 for the Gallilean transform case. This gives

$$\left(\frac{\partial^2}{\partial x^2}\right)_t = \left(\frac{\partial^2}{\partial x'^2}\right)_{t'}$$

(check!) as expected. Substituting an arbitrary transformation is just as easy, though. Good luck.